FREQUENTLY HYPERCYCLIC C-DISTRIBUTION SEMIGROUPS AND THEIR GENERALIZATIONS

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Abstract. In this paper, we introduce the notions of f-frequent hypercyclicity and $\mathcal{F}$-hypercyclicity for C-distribution semigroups in separable Fréchet spaces. We particularly analyze the classes of q-frequently hypercyclic C-distribution semigroups ($q \geq 1$) and frequently hypercyclic C-distribution semigroups, providing a great number of illustrative examples.

1. Introduction and preliminaries

The notion of a frequently hypercyclic linear continuous operator on a separable Fréchet space was introduced by F. Bayart and S. Grivaux in 2006 [2]. The general notion of (m_k)-hypercyclicity for linear continuous operators was introduced by F. Bayart and É. Matheron [3] in 2009, while some special cases of (m_k)-hypercyclicity, like q-frequent hypercyclicity ($q \in \mathbb{N}$), were analyzed by M. Gupta and A. Mundayadan in [16]. Within the field of linear topological dynamics, the notion of $\mathcal{F}$-hypercyclicity, where $\mathcal{F}$ is a Furstenberg family, was introduced for the first time by S. Shkarin in 2009 [27]; further contributions were given by A. Bonilla and K.-G. Grosse-Erdmann [6]. The notion of $\mathcal{F}$-hypercyclicity for linear not necessarily continuous operators has been recently introduced by the author in [21]. For more details on the subject, we refer the reader to $[1, 4, 8, 15, 25]$ and references cited therein.

On the other hand, the notion of a frequently hypercyclic strongly continuous semigroup on a separable Banach space was introduced by E. M. Mangino and A. Peris in 2011 [23]. Frequently hypercyclic translation semigroups on weighted function spaces were further investigated by E. M. Mangino and M. Murillo-Arcila in [24]. Frequent hypercyclicity and various generalizations of this concept for single operators and semigroups of operators are still very active field of research, full of open unsolved problems.

Hypercyclicity of C-regularized semigroups, distribution semigroups and unbounded linear operators in Banach spaces was analyzed by R. deLaubenfels, H. Emamirad and K.-G. Grosse-Erdmann in 2003 [10]. The non-existence of an appropriate reference which treats the frequent hypercyclicity of C-regularized semigroups and distribution semigroups strongly influenced us to write this paper.

We work in the setting of separable infinite-dimensional Fréchet spaces, considering general classes of C-distribution semigroups and fractionally integrated C-semigroups [18, 19]; here, we would like to point out that our results seem to be new even for strongly continuous semigroups of operators in Fréchet spaces. In contrast to the investigations of frequently hypercyclic strongly continuous semigroups of operators that are carried out so far, the notion of Pettis integrability does not play any significant role in our approach, which is primarly oriented for giving some new applications in the qualitative analysis of solutions of abstract ill-posed differential equations of first order. The notion of a q-frequently hypercyclic strongly continuous semigroup, where $q \geq 1$, has been recently introduced and systematically analyzed in our joint paper with B. Chaouchi, S. Pilipović and D. Velinov [7]; the notion of f-frequent hypercyclicity, introduced here for the first time as a continuous counterpart of (m_k)-hypercyclicity, seems to be not considered elsewhere even for strongly continuous semigroups of operators in Banach spaces. Albeit we analyze the general class of C-distribution semigroups, providing also some examples of frequently hypercyclic integrated semigroups, almost all structural results of ours are stated for the class of global C-regularized semigroups (for certain difficuties we have met in our exploration of frequently hypercyclic fractionally integrated C-semigroups, we refer the reader to Remark 2.6).

Without any doubt, our main theoretical result is Theorem 2.5, which can be called f-Frequent Hypercyclicity Criterion for C-Regularized Semigroups. In Theorem 2.8, we state Upper Frequent Hypercyclicity Criterion for C-Regularized Semigroups (this result seems to be new even for strongly continuous semigroups in Banach spaces, as well). From the point of view of possible applications, Theorem 2.10, in which we reconsider the spectral criterions esatablished by S. El Mourchid [13, Theorem 2.1] and E. M. Mangino, A. Peris [23, Corollary 2.3], and Theorem 2.12, in which we reconsider the famous Desch-Schappacher-Webb criterion for chaos of strongly continuous semigroups [11, Theorem 3.1], are most important; both theorems are consequences of Theorem 2.5. In Example 2.14, we revisit [18, Subsection 3.1.4] and prove that all examined C-regularized semigroups and integrated semigroups, including corresponding single linear operators, are frequently hypercyclic (we already know that these semigroups and operators are topologically mixing or chaotic in a certain sense).

We use the standard notation throughout the paper. By E we denote a separable infinite-dimensional Fréchet space (real or complex). We assume that the topology of E is induced by the fundamental system $(p_n)_{n \in \mathbb{N}}$ of increasing seminorms. If Y is also a Fréchet space, over the same field of scalars $\mathbb{K}$ as E, then by $L(E, Y)$ we denote the space consisting of all continuous linear mappings from E into Y. The translation

invariant metric $d : E \times E \to [0, \infty)$, defined by

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x - y)}{1 + p_n(x - y)}, \ x, \ y \in E,$$

satisfies, among many other properties, the following ones: $d(x + u, y + v) \le d(x, y) + d(u, v)$ and $d(cx, cy) \le (|c| + 1)d(x, y)$, $c \in \mathbb{K}$, x, y, u, $v \in X$. Set $L(x, \epsilon) := \{y \in X : d(x, y) < \epsilon\}$ and $L_n(x, \epsilon) := \{y \in X : p_n(x - y) < \epsilon\}$ ($n \in \mathbb{N}$, $\epsilon > 0$, $x \in X$). By E^* we denote the dual space of E. For a closed linear operator T on E, we denote by $D(T)$, $R(T)$, $N(T)$, $\rho(T)$ and $\sigma_p(T)$ its domain, range, kernel, resolvent set and point spectrum, respectively. If $\tilde{E}$ is a linear subspace of E, then the part of T in $\tilde{E}$, $T_{|\tilde{E}}$ shortly, is defined through $T_{|\tilde{E}} := \{(x, y) \in T : x, \ y \in \tilde{E}\}$. Set $D_\infty(T) := \bigcap_{k \in \mathbb{N}} D(T^k)$. We will always assume henceforth that $C \in L(E)$ and C is injective. Put $p_C(x) := p(C^{-1}x)$, $p \in \circledast$, $x \in R(C)$. Then $p_C(\cdot)$ is a seminorm on $R(C)$ and the calibration $(p_C)_{p \in \circledast}$ induces a Fréchet topology on $R(C)$; we denote this space by $[R(C)]_{\circledast}$. If T^k is closed for any $k \in \mathbb{N}$, then the space $C(D(T^k))$, equipped with the following family of seminorms $p_{k,n}(Cx) := p_n(x) + p_n(Tx) + \cdots + p_n(T^k x)$, $x \in D(T^k)$, is a Fréchet one ($n \in \mathbb{N}$). This space will be denoted by $[C(D(T^k))]$. For any $s \in \mathbb{R}$, we define $\lfloor s \rfloor := \sup\{l \in \mathbb{Z} : s \ge l\}$ and $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \le l\}$.

Let us recall that a series $\sum_{n=1}^{\infty} x_n$ in E is called unconditionally convergent iff for every permutation σ of $\mathbb{N}$, the series $\sum_{n=1}^{\infty} x_{\sigma(n)}$ is convergent; it is well known that the absolute convergence of $\sum_{n=1}^{\infty} x_n$ (i.e., the convergence of $\sum_{n=1}^{\infty} p_l(x_n)$ for all $l \in \mathbb{N}$) implies its unconditional convergence (see [5] and references cited therein for further information on the subject).

The Schwartz space of rapidly decreasing functions $\mathcal{S}$ with values in the field $\mathbb{K}$ is defined by the following system of seminorms $p_{m,n}(\psi) := \sup_{x \in \mathbb{R}} |x^m \psi^{(n)}(x)|$, $\psi \in \mathcal{S}$, m, $n \in \mathbb{N}_0$. Similarly, we use notation $\mathcal{D} = C_0^\infty(\mathbb{R})$ and $\mathcal{E} = C^\infty(\mathbb{R})$. If $\emptyset \ne \Omega \subseteq \mathbb{R}$, then the symbol $\mathcal{D}_\Omega$ denotes the subspace of $\mathcal{D}$ consisting of those functions $\varphi \in \mathcal{D}$ for which $\mathrm{supp}(\varphi) \subseteq \Omega$; $\mathcal{D}_0 \equiv \mathcal{D}_{[0,\infty)}$. The spaces $\mathcal{D}'(E) := L(\mathcal{D}, E)$, $\mathcal{E}'(E) := L(\mathcal{E}, E)$ and $\mathcal{S}'(E) := L(\mathcal{S}, E)$ are topologized in the usual way; the symbols $\mathcal{D}'_\Omega(E)$, $\mathcal{E}'_\Omega(E)$ and $\mathcal{S}'_\Omega(E)$ denote their subspaces containing E-valued distributions whose supports are contained in Ω; $\mathcal{D}'_0(E) \equiv \mathcal{D}'_{[0,\infty)}(E)$, $\mathcal{E}'_0(E) \equiv \mathcal{E}'_{[0,\infty)}(E)$, $\mathcal{S}'_0(E) \equiv \mathcal{S}'_{[0,\infty)}(E)$ (see [26] for more details). By δ_t we denote the Dirac distribution centered at point t ($t \in \mathbb{R}$). If $\varphi, \psi : \mathbb{R} \to \mathbb{C}$ are measurable functions, then we define $\varphi *_0 \psi(t) := \int_0^t \varphi(t - s)\psi(s) \, ds$, $t \in \mathbb{R}$. The convolution of vector-valued distributions will be taken in the sense of [22, Proposition 1.1].

Let $C \in L(E)$ be an injective operator, and let $\mathcal{G} \in \mathcal{D}'^*_0(L(E))$ satisfy $C\mathcal{G} = \mathcal{G}C$. We refer the reader to [20] for the notion of a (dense) C-distribution semigroup $\mathcal{G}$ and the closed linear operator $G(T)$, where $T \in \mathcal{E}'_0$. We define the (infinitesimal) generator of a (C-DS) $\mathcal{G}$ by $A := G(-\delta')$. If S, $T \in \mathcal{E}'_0$, $\varphi \in \mathcal{D}_0$, $\psi \in \mathcal{D}$ and $x \in E$, then we know that the following holds:

A1. $G(S)G(T) \subseteq G(S * T)$ with $D(G(S)G(T)) = D(G(S * T)) \cap D(G(T))$, and $G(S) + G(T) \subseteq G(S + T)$.

We denote by $D(\mathcal{G})$ the set consisting of those elements $x \in E$ for which $x \in$

$D(G(\delta_t))$, $t \geq 0$ and the mapping $t \mapsto G(\delta_t)x$, $t \geq 0$ is continuous. By **A1.**, we have that

$$D\big(G(\delta_s)G(\delta_t)\big) = D\big(G(\delta_s * \delta_t)\big) \cap D\big(G(\delta_t)\big) = D\big(G(\delta_{t+s})\big) \cap D\big(G(\delta_t)\big), \ t, \ s \geq 0,$$

which clearly implies $G(\delta_t)(D(\mathcal{G})) \subseteq D(\mathcal{G})$, $t \geq 0$ and

$$G(\delta_s)G(\delta_t)x = G(\delta_{t+s})x, \quad t, \ s \geq 0, \ x \in D(\mathcal{G}). \tag{1}$$

The definitions of of an (exponentially equicontinuous) α-times integrated C-semigroup $(S_\alpha(t))_{t \geq 0}$, its subgenerators and the integral generator are well known (see e.g. [20] and the references cited therein). If $\alpha = 0$, then $(S_0(t))_{t \geq 0}$ is also said to be a C-regularized semigroup with subgenerator A; in this case, we have the following simple functional equation: $S_0(t)S_0(s) = S_0(t + s)C$, t, $s \geq 0$.

Let A be a closed linear operator on E. Denote by $Z_1(A)$ the space consisting of those elements $x \in E$ for which there exists a unique continuous mapping $u : [0, \infty) \to E$ satisfying $\int_0^t u(s, x)\, ds \in D(A)$ and $A \int_0^t u(s, x)\, ds = u(t, x) - x$, $t \geq 0$; cf. [18–20] for more details on the subject.

Suppose that A is a subgenerator of an α-times integrated C-semigroup $(S_\alpha(t))_{t \geq 0}$. Then A is a subgenerator of a $\lceil \alpha \rceil$-integrated C-semigroup $(S_{\lceil \alpha \rceil}(t) \equiv (g_{\lceil \alpha \rceil - \alpha} *_0 S_\alpha)(t))_{t \geq 0}$. Set

$$\mathcal{G}(\varphi)x := (-1)^{\lceil \alpha \rceil} \int_0^\infty \varphi^{(\lceil \alpha \rceil)}(t) S_{\lceil \alpha \rceil}(t)x\, dt, \quad \varphi \in \mathcal{D}, \ x \in E \tag{2}$$

and $\qquad G(\delta_t)x := \dfrac{d^{\lceil \alpha \rceil}}{dt^{\lceil \alpha \rceil}} C^{-1} S_{\lceil \alpha \rceil}(t)x, \quad t \geq 0, \ x \in Z_1(A).$

If $C_1 \in L(X)$ is another injective operator with $C_1 A \subseteq A C_1$, $\gamma \geq 0$ and A is a subgenerator (the integral generator) of a global γ-times integrated C_1-semigroup $(S^\gamma(t))_{t \geq 0}$, then $(S_\alpha(t))_{t \geq 0}$ and $(S^\gamma(t))_{t \geq 0}$ share the same (subspace) f-frequently hypercyclic properties defined below.

We refer the reader to [18–20] for further information concerning C-distribution semigroups. The notion of exponentially equicontinuous, analytic fractionally integrated C-semigroups will be taken in a broad sense of [19, Definition 2.2.1(i)], while the notion of an entire C-regularized group will be taken in the sense of [19, Definition 2.2.9].

1.1 Lower and upper densities

First of all, we need to recall the following definitions.

DEFINITION 1.1 ([21]). Let $(T_n)_{n \in \mathbb{N}}$ be a sequence of linear operators acting between the spaces X and Y, let T be a linear operator on X, and let $x \in X$. Suppose that $\mathcal{F} \in P(P(\mathbb{N}))$ and $\mathcal{F} \neq \emptyset$. Then we say that:
(i) x is an $\mathcal{F}$-hypercyclic element of the sequence $(T_n)_{n \in \mathbb{N}}$ iff $x \in \bigcap_{n \in \mathbb{N}} D(T_n)$ and for each open non-empty subset V of Y we have that $S(x, V) := \{n \in \mathbb{N} : T_n x \in V\} \in \mathcal{F}$; $(T_n)_{n \in \mathbb{N}}$ is said to be $\mathcal{F}$-hypercyclic iff there exists an $\mathcal{F}$-hypercyclic element of $(T_n)_{n \in \mathbb{N}}$;

(ii) T is $\mathcal{F}$-hypercyclic iff the sequence $(T^n)_{n \in \mathbb{N}}$ is $\mathcal{F}$-hypercyclic; $x \in D_\infty(T)$ is said to be an $\mathcal{F}$-hypercyclic element of T iff x is an $\mathcal{F}$-hypercyclic element of the sequence $(T^n)_{n \in \mathbb{N}}$.

DEFINITION 1.2. Let $q \in [1, \infty)$, let $A \subseteq \mathbb{N}$, and let (m_n) be an increasing sequence in $[1, \infty)$. Then:

(i) The lower q-density of A, denoted by $\underline{d}_q(A)$, is defined through:

$$\underline{d}_q(A) := \liminf_{n \to \infty} \frac{|A \cap [1, n^q]|}{n}.$$

(ii) The upper q-density of A, denoted by $\overline{d}_q(A)$, is defined through:

$$\overline{d}_q(A) := \limsup_{n \to \infty} \frac{|A \cap [1, n^q]|}{n}.$$

(iii) The lower (m_n)-density of A, denoted by $\underline{d}_{m_n}(A)$, is defined through:

$$\underline{d}_{m_n}(A) := \liminf_{n \to \infty} \frac{|A \cap [1, m_n]|}{n}.$$

(iv) The upper (m_n)-density of A, denoted by $\overline{d}_{m_n}(A)$, is defined through:

$$\overline{d}_{m_n}(A) := \limsup_{n \to \infty} \frac{|A \cap [1, m_n]|}{n}.$$

Assume that $q \in [1, \infty)$ and (m_n) is an increasing sequence in $[1, \infty)$. Consider the notion introduced in Definition 1.1 with: (i) $\mathcal{F} = \{A \subseteq \mathbb{N} : \underline{d}(A) > 0\}$, (ii) $\mathcal{F} = \{A \subseteq \mathbb{N} : \underline{d}_q(A) > 0\}$, (iii) $\mathcal{F} = \{A \subseteq \mathbb{N} : \underline{d}_{m_n}(A) > 0\}$; then we say that $(T_n)_{n \in \mathbb{N}}$ (T, x) is frequently hypercyclic, q-frequently hypercyclic and l-(m_n)-hypercyclic, respectively.

Denote by $m(\cdot)$ the Lebesgue measure on $[0, \infty)$. We would like to propose the following definition:

DEFINITION 1.3. Let $q \in [1, \infty)$, let $A \subseteq [0, \infty)$, and let $f : [0, \infty) \to [1, \infty)$ be an increasing mapping. Then:

(i) The lower qc-density of A, denoted by $\underline{d}_{qc}(A)$, is defined through:

$$\underline{d}_{qc}(A) := \liminf_{t \to \infty} \frac{m(A \cap [0, t^q])}{t}.$$

(ii) The upper qc-density of A, denoted by $\overline{d}_{qc}(A)$, is defined through:

$$\overline{d}_{qc}(A) := \limsup_{t \to \infty} \frac{m(A \cap [0, t^q])}{t}.$$

(iii) The lower f-density of A, denoted by $\underline{d}_f(A)$, is defined through:

$$\underline{d}_f(A) := \liminf_{t \to \infty} \frac{m(A \cap [0, f(t)])}{t}.$$

(iv) The upper f-density of A, denoted by $\overline{d}_f(A)$, is defined through:

$$\overline{d}_f(A) := \limsup_{t \to \infty} \frac{m(A \cap [0, f(t)])}{t}.$$

It is clear that Definition 1.3 provides continuous analogues of the notion introduced in Definition 1.2, which have been analyzed in [21, Section 2] in more detail. For the sake of brevity and better exposition, we will skip all related details about possibilities to transfer the results established in [21] for continuous lower and upper densities.

2. Generalized frequent hypercyclicity for C-distribution semigroups and fractionally integrated C-semigroups

Let $P([0, \infty))$ denote the power set of $[0, \infty)$.

DEFINITION 2.1. Let $\mathcal{G}$ be a C-distribution semigroup, and let $x \in D(\mathcal{G})$. Suppose that $\mathcal{F} \in P(P([0, \infty)))$ and $\mathcal{F} \neq \emptyset$. Then we say that x is an $\mathcal{F}$-hypercyclic element of $\mathcal{G}$ iff for each open non-empty subset V of E we have $S(x, V) := \{t \geq 0 : G(\delta_t)x \in V\} \in \mathcal{F}$; $\mathcal{G}$ is said to be $\mathcal{F}$-hypercyclic iff there exists an $\mathcal{F}$-hypercyclic element of $\mathcal{G}$.

The notion introduced in the following definition is a special case of the notion introduced above, with $\mathcal{F}$ being the collection of all non-empty subsets A of $[0, \infty)$ such that the lower qc-density of A, the upper qc-density of A, the lower f-density of A or the upper f-density of A is positive:

DEFINITION 2.2. Let $q \in [1, \infty)$, and let $f : [0, \infty) \to [1, \infty)$ be an increasing mapping. Suppose that $\mathcal{G}$ is a C-distribution semigroup. Then we say that:
(i) $\mathcal{G}$ is q-frequently hypercyclic iff there exists $x \in D(\mathcal{G})$ such that for each open non-empty subset V of E we have $\underline{d}_{qc}(\{t \geq 0 : G(\delta_t)x \in V\}) > 0$;

(ii) $\mathcal{G}$ is upper q-frequently hypercyclic iff there exists $x \in D(\mathcal{G})$ such that for each open non-empty subset V of E we have $\overline{d}_{qc}(\{t \geq 0 : G(\delta_t)x \in V\}) > 0$;

(iii) $\mathcal{G}$ is f-frequently hypercyclic iff there exists $x \in D(\mathcal{G})$ such that for each open non-empty subset V of E we have $\underline{d}_f(\{t \geq 0 : G(\delta_t)x \in V\}) > 0$;

(iv) $\mathcal{G}$ is upper f-frequently hypercyclic iff there exists $x \in D(\mathcal{G})$ such that for each open non-empty subset V of E we have $\underline{d}_f(\{t \geq 0 : G(\delta_t)x \in V\}) > 0$.

It seems natural to reformulate the notion introduced in the previous two definitions for fractionally integrated C-semigroups:

DEFINITION 2.3. Suppose that A is a subgenerator of a global α-times integrated C-semigroup $(S_\alpha(t))_{t \geq 0}$ for some $\alpha \geq 0$. Let $\mathcal{F} \in P(P([0, \infty)))$ and $\mathcal{F} \neq \emptyset$. Then we say that an element $x \in Z_1(A)$ is an $\mathcal{F}$-hypercyclic element of $(S_\alpha(t))_{t \geq 0}$ iff x is an $\mathcal{F}$-hypercyclic element of the induced C-distribution semigroup $\mathcal{G}$ defined through (2); $(S_\alpha(t))_{t \geq 0}$ is said to be $\mathcal{F}$-hypercyclic iff $\mathcal{G}$ is said to be $\mathcal{F}$-hypercyclic.

DEFINITION 2.4. Suppose that A is a subgenerator of a global α-times integrated C-semigroup $(S_\alpha(t))_{t \geq 0}$ for some $\alpha \geq 0$. Let $q \in [1, \infty)$, and let $f : [0, \infty) \to [1, \infty)$ be an increasing mapping. Then it is said that $(S_\alpha(t))_{t \geq 0}$ is q-frequently hypercyclic (upper

q-frequently hypercyclic, f-frequently hypercyclic, upper f-frequently hypercyclic) iff the induced C-distribution semigroup $\mathcal{G}$, defined through (2), is such.

As mentioned in the introductory part, the following result can be viewed as f-Frequent Hypercyclicity Criterion for C-Regularized Semigroups:

THEOREM 2.5. *Suppose that A is a subgenerator of a global C-regularized semigroup $(S_0(t))_{t\geq 0}$ on E and $f : [0,\infty) \to [1,\infty)$ is an increasing mapping. Set $T(t)x := C^{-1}S_0(t)x$, $t \geq 0$, $x \in Z_1(A)$ and $m_k := f(k)$, $k \in \mathbb{N}$. Suppose that there are a number $t_0 > 0$, a dense subset E_0 of E and mappings $S_n : E_0 \to R(C)$ $(n \in \mathbb{N})$ such that the following conditions hold for all $y \in E_0$:*

(i) The series $\sum\limits_{n=1}^{k} T(t_0\lfloor m_k\rfloor)S_{\lfloor m_{k-n}\rfloor}y$ converges unconditionally, uniformly in $k \in \mathbb{N}$.

(ii) The series $\sum\limits_{n=1}^{\infty} T(t_0\lfloor m_k\rfloor)S_{\lfloor m_{k+n}\rfloor}y$ converges unconditionally, uniformly in $k \in \mathbb{N}$.

(iii) The series $\sum\limits_{n=1}^{\infty} S_{\lfloor m_n\rfloor}y$ converges unconditionally, uniformly in $n \in \mathbb{N}$.

(iv) $\lim\limits_{n\to\infty} T(t_0\lfloor m_n\rfloor)S_{\lfloor m_n\rfloor}y = y$.

(v) $R(C)$ is dense in E.
Then $(S_0(t))_{t\geq 0}$ is f-frequently hypercyclic and the operator $T(t_0)$ is l-(m_k)-frequently hypercyclic.

Proof. Without loss of generality, we may assume that $t_0 = 1$. It is clear that (m_k) is an increasing sequence in $[1,\infty)$. Define the sequence of operators $(T_n)_{n\in\mathbb{N}} \subseteq L([R(C)], E)$ by $T_nx := T(n)x$, $n \in \mathbb{N}$, $x \in R(C)$. Due to (1), we get that $T_nx = T(1)^nx$ for $x \in R(C)$. Then the prescribed assumptions (i)-(iv) in combination with [21, Theorem 3.1] imply that the sequence $(T_n)_{n\in\mathbb{N}}$ is l-(m_k)-frequently hypercyclic, which means that there exists an element $x = Cy \in R(C)$, for some $y \in E$, satisfying that for each open non-empty subset V' of E there exists an increasing sequence (k_n) of positive integers such that the interval $[1, f(k_n)]$ contains at least k_nc elements of set $\{k \in \mathbb{N} : T_kCy = S_0(k)y \in V'\}$. Since $T_n \subseteq T(1)^n$, the above clearly implies that the operator $T(1)$ is l-(m_k)-frequently hypercyclic with $x = Cy$ being its l-(m_k)-frequently hypercyclic vector. We will prove that $Cx = C^2y$ is an f-frequently hypercyclic vector for $(S_0(t))_{t\geq 0}$, i.e., that for each open non-empty subset V of E we have $\underline{d}_f(\{t \geq 0 : T(t)Cx = S_0(t)Cy \in V\}) > 0$; see also the proof of [23, Proposition 2.1]. Let such a set V be given. Then, due to our assumption (v), there exist an element $z \in E$ and a positive integer $n \in \mathbb{N}$ such that $L_n(Cz, \epsilon) \subseteq V$. By the local equicontinuity of $(S_0(t))_{t\geq 0}$ and the continuity of C, we get that there exist an integer $m \in \mathbb{N}$ and a positive constant $c > 1$ such that $p_n(Cx) \leq cp_m(x)$, $x \in E$ and

$$
\begin{aligned}
p_n\big(S_0(k+\delta)Cy - S_0(k)Cy\big) &\leq p_n\big(S_0(k+\delta)Cy - Cz\big) + p_n\big(S_0(k)Cy - Cz\big) \\
&\leq p_n\big(S_0(\delta)[S_0(k)y - z]\big) + p_n\big(S_0(\delta)z - Cz\big) + p_n\big(S_0(k)Cy - Cz\big) \\
&\leq cp_m\big(S_0(k)y - z\big) + p_n\big(S_0(\delta)z - Cz\big), \quad \delta \in [0,1]. \tag{3}
\end{aligned}
$$

Set $V' := L_m(z, \epsilon/3c)$. Then, by the foregoing, we know that there exists an increasing sequence (k_n) of positive integers such that the interval $[1, f(k_n)]$ contains at least $k_n c$ elements of set $\mathrm{A} := \{k \in \mathbb{N} : T_k Cy = S_0(k)y \in V'\}$. For any $k \in \mathrm{A}$, we have $p_m(S_0(k)y - z) < \epsilon/3c$; further on, (3) yields that there exists a positive constant $\delta_0 > 0$ such that $p_n(S_0(k+\delta)Cy - S_0(k)Cy) < 2\epsilon/3$ for all $\delta \in [0, \delta_0]$. This implies

$$p_n\big(S_0(k+\delta)Cy - Cz\big) \leq p_n\big(S_0(k+\delta)Cy - S_0(k)Cy\big) + p_n\big(S_0(k)Cy - Cz\big) < \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

for any $k \in \mathrm{A}$ and $\delta \in [0, \delta_0]$. By virtue of this, we conclude that $S_0(k+\delta)Cy \in V$ for any $k \in \mathrm{A}$ and $\delta \in [0, \delta_0]$, finishing the proof of theorem in a routine manner. $\square$

Plugging $f(t) := t^q + 1$, $t \geq 0$ $(q \geq 1)$, we obtain a sufficient condition for q-frequent hypercyclicity of $(S_0(t))_{t\geq 0}$ and $T(t_0)$.

REMARK 2.6. Consider the situation of Theorem 2.5 with A being a subgenerator of a global α-times integrated C-semigroup $(S_\alpha(t))_{t\geq 0}$ on X, $n \geq \lceil \alpha \rceil$ and the Fréchet space $[C(D(A^n))]$ being separable. It is well known that $C(D(A^n)) \subseteq Z_1(A)$; if $x = Cy \in C(D(A^n))$, then for every $t \geq 0$,

$$G(\delta_t)x = \frac{d^n}{dt^n}C^{-1}S_n(t)x = \frac{d^n}{dt^n}S_n(t)y = S_n(t)A^n y + \sum_{i=0}^{n-1} \frac{t^{n-i-1}}{(n-i-1)!}CA^{n-1-i}y.$$

Furthermore, for every $t \geq 0$, the mapping $G(\delta_t) : [C(D(A^n))] \to X$ is linear and continuous as well as the operator family $(G(\delta_t))_{t\geq 0} \subseteq L([C(D(A^n))], X)$ is strongly continuous; see e.g. the proof of [9, Theorem 5.4]. But, it is not clear how to prove that the l-(m_k)-frequent hypercyclicity of a single operator $G(\delta_{t_0})_{|C(D(A^n))}$, for some $t_0 > 0$, implies the l-(m_k)-frequent hypercyclicity of $(S_n(t))_{t\geq 0}$ because an analogue of the estimate (3) seems to be not attainable for integrated C-semigroups. The interested reader may try to prove an analogue of [18, Theorem 3.1.32] for l-(m_k)-frequent hypercyclicity.

Suppose now that $\mathcal{F} \in P(P(\mathbb{N}))$ and $\mathcal{F} \neq \emptyset$. If $\mathcal{F}$ satisfies the following property:
(I) $A \in \mathcal{F}$ and $A \subseteq B$ imply $B \in \mathcal{F}$,
then it is said that $\mathcal{F}$ is a Furstenberg family; a proper Furstenberg family $\mathcal{F}$ is any Furstenberg family satisfying that $\emptyset \notin \mathcal{F}$. See [14] for more details.

From the proof of Theorem 2.5, we may deduce the following.

PROPOSITION 2.7. *Let $\mathcal{F}$ be a Furstenberg family. Suppose that A is a subgenerator of a global C-regularized semigroup $(S_0(t))_{t\geq 0}$ on E and $T(t)x := C^{-1}S_0(t)x$, $t \geq 0$, $x \in Z_1(A)$. If $R(C)$ is dense in E, $t_0 > 0$ and $x \in Z_1(A)$ is an $\mathcal{F}$-hypercyclic element of $T(t_0)$, then x is an $\mathcal{F}'$-hypercyclic element of $(S_0(t))_{t\geq 0}$, where*

$$\mathcal{F}' = \left\{ B \subseteq [0, \infty) : (\exists A \in \mathcal{F})\,(\exists \delta_0 > 0)\, \bigcup_{k \in A}[k, k+\delta_0] \subseteq B \right\}. \tag{4}$$

An upper Furstenberg family is any proper Furstenberg family $\mathcal{F}$ satisfying the following two conditions:
(II) There are a set D and a countable set M such that $\mathcal{F} = \bigcup_{\delta \in D} \bigcap_{\nu \in M} \mathcal{F}_{\delta, \nu}$, where for each $\delta \in D$ and $\nu \in M$ the following holds: If $A \in \mathcal{F}_{\delta, \nu}$, then there exists a finite subset $F \subseteq \mathbb{N}$ such that the implication $A \cap F \subseteq B \Rightarrow B \in \mathcal{F}_{\delta, \nu}$ holds true.

(III) If $A \in \mathcal{F}$, then there exists $\delta \in D$ such that, for every $n \in \mathbb{N}$, we have $A - n \equiv \{k - n : k \in A, \ k > n\} \in \mathcal{F}_\delta$, where $\mathcal{F}_\delta \equiv \bigcap_{\nu \in M} \mathcal{F}_{\delta,\nu}$.

Appealing to [6, Theorem 22] in place of [21, Theorem 3.1], and repeating almost literally the arguments given in the proof of Theorem 2.5, we may deduce the following result.

THEOREM 2.8. *Suppose that $\mathcal{F} = \bigcup_{\delta \in D} \bigcap_{\nu \in M} \mathcal{F}_{\delta,\nu}$ is an upper Furstenberg family and A is a subgenerator of a global C-regularized semigroup $(S_0(t))_{t \geq 0}$ on E. Set $T(t)x := C^{-1} S_0(t)x$, $t \geq 0$, $x \in Z_1(A)$. Suppose that there are a number $t_0 > 0$, two dense subsets E_0' and E_0'' of E and mappings $S_n : E_0'' \to R(C)$ $(n \in \mathbb{N})$ such that for any $y \in E_0''$ and $\epsilon > 0$ there exist $A \in \mathcal{F}$ and $\delta \in D$ such that:*
(i) For every $x \in E_0'$, there exists some $B \in \mathcal{F}_\delta$, $B \subseteq A$ such that, for every $n \in B$, one has $S_0(t_0 n)x \in L(0, \epsilon)$.

(ii) The series $\sum_{n \in A} S_n y$ converges.

(iii) For every $m \in A$, we have $T(mt_0) \sum_{n \in A} S_n y - y \in L(0, \epsilon)$.

(iv) $R(C)$ is dense in E.
Then the operator $T(t_0)$ is $\mathcal{F}$-hypercyclic and $(S_0(t))_{t \geq 0}$ is $\mathcal{F}'$-hypercyclic, where $\mathcal{F}'$ is given by (4).

REMARK 2.9. Collection of all non-empty subsets $A \subseteq [0, \infty)$ for which $\overline{d}_{qc}(A) > 0$ forms an upper Furstenberg family ([6,21]), so that Theorem 2.8 with $f(t) = t^q + 1$, $t \geq 0$ $(q \geq 1)$ gives a sufficient condition for the upper q-frequent hypercyclicity of $(S_0(t))_{t \geq 0}$ and $T(t_0)$. It can be simply proved that the validity of condition $\limsup_{t \to \infty} \frac{f(t)}{t} > 0$ for an increasing function $f : [0, \infty) \to [1, \infty)$ implies that the collection of all non-empty subsets $A \subseteq [0, \infty)$ such that $\overline{d}_f(A) > 0$ forms an upper Furstenberg family, as well.

We continue by stating two intriguing consequences of Theorem 2.5. The first one is motivated by the well-known results of S. El Mourchid [13, Theorem 2.1] and E. M. Mangino, A. Peris [23, Corollary 2.3]; see also [18, Theorem 3.1.40].

THEOREM 2.10. *Let $t_0 > 0$, let $\mathbb{K} = \mathbb{C}$, and let A be a subgenerator of a global C-regularized semigroup $(S_0(t))_{t \geq 0}$ on E. Suppose that $R(C)$ is dense in E. Set $T(t)x := C^{-1} S_0(t)x$, $t \geq 0$, $x \in Z_1(A)$.*
(i) Assume that there exists a family $(f_j)_{j \in \Gamma}$ of locally bounded measurable mappings $f_j : I_j \to E$ such that I_j is an interval in $\mathbb{R}$, $A f_j(t) = it f_j(t)$ for every $t \in I_j$, $j \in \Gamma$ and $\mathrm{span}\{f_j(t) : j \in \Gamma, \ t \in I_j\}$ is dense in E. If $f_j \in C^2(I_j : X)$ for every $j \in \Gamma$, then $(S_0(t))_{t \geq 0}$ is frequently hypercyclic and each single operator $T(t_0)$ is frequently hypercyclic.

(ii) Assume that there exists a family $(f_j)_{j \in \Gamma}$ of twice continuously differentiable mappings $f_j : I_j \to E$ such that I_j is an interval in $\mathbb{R}$ and $A f_j(t) = it f_j(t)$ for every $t \in I_j$, $j \in \Gamma$. Set $\tilde{E} := \overline{\mathrm{span}\{f_j(t) : j \in \Gamma, \ t \in I_j\}}$. Then $A_{|\tilde{E}}$ is a subgenerator of a global $C_{|\tilde{E}}$-regularized semigroup $(S_0(t)_{|\tilde{E}})_{t \geq 0}$ on $\tilde{E}$, $(S_0(t)_{|\tilde{E}})_{t \geq 0}$ is frequently hypercyclic in $\tilde{E}$ and the operator $T(t_0)_{|\tilde{E}}$ is frequently hypercyclic in $\tilde{E}$.

Proof. Consider first the statement (i). Arguing as in the Banach space case [23, Corollary 2.3], we get that there exists a family $(g_j)_{j \in \Lambda}$ of functions $g_j \in C^2(\mathbb{R} : E)$ with compact support such that $Ag_j(t) = itg_j(t)$ for every $t \in \mathbb{R}$, $j \in \Lambda$ and span$\{g_j(t) : j \in \Lambda, \ t \in \mathbb{R}\}$ is dense in E. For every $\lambda \in \Lambda$ and $r \in \mathbb{R}$, set $\psi_{r,\lambda} := \int_{-\infty}^{\infty} e^{-irs} g_\lambda(s) \, ds$. Then we have

$$T(t)\psi_{r,\lambda} = \psi_{r-t,\lambda}, \quad t \geq 0, \ r \in \mathbb{R}, \ \lambda \in \Lambda \tag{5}$$

and the part (i) follows by applying Theorem 2.5 with the sequence $m_k := k$ $(k \in \mathbb{N})$, $E_0 := C(\text{span}\{g_j(t) : j \in \Lambda, \ t \in \mathbb{R}\})$ and the operator $S_n : E_0 \to R(C)$ given by $S_n(C\psi_{r,\lambda}) := C\psi_{t_0 n + r,\lambda}$ $(n \in \mathbb{N}, \ r \in \mathbb{R}, \ \lambda \in \Lambda)$ and after that linearly extended to E_0 in the obvious way; here, it is only worth noting that the conditions (i)-(iii) follow from (5) and the fact that the series $\sum_{n=1}^{\infty} \psi_{t_0 n + r,\lambda}$ and $\sum_{n=1}^{\infty} \psi_{-t_0 n + r,\lambda}$ converge absolutely (and therefore, unconditionally) since for each seminorm $p_l(\cdot)$, where $l \in \mathbb{N}$, there exists a finite constant $c_l > 0$ such that $p_l(\psi_{t_0 n + r,\lambda}) + p_l(\psi_{-t_0 n + r,\lambda}) \leq c_l n^{-2}$, $n \in \mathbb{N}$ $(r \in \mathbb{R}, \ \lambda \in \Lambda)$. This can be seen by applying integration by parts twice, as in the proof of [15, Lemma 9.23(b)].

For the proof of (ii), it is enough to observe that an elementary argumentation shows that $A_{|\tilde{E}}$ is a subgenerator of a global $C_{|\tilde{E}}$-regularized semigroup $(S_0(t)_{|\tilde{E}})_{t \geq 0}$ on $\tilde{E}$. Then we can apply (i) to finish the proof. $\qquad\square$

The following application of Theorem 2.10 is quite illustrative $(C = I)$.

EXAMPLE 2.11. Consider the operator $A := d/dt$, acting with maximal domain in the Banach space $E := BUC(\mathbb{R})$, consisting of all bounded uniformly continuous functions. Then $\sigma_p(A) = i\mathbb{R}$ and $Ae^{\lambda \cdot} = \lambda e^{\lambda \cdot}$, $\lambda \in i\mathbb{R}$. It is well-known that the space $\tilde{E} := \overline{\text{span}\{e^{\lambda \cdot} : \lambda \in i\mathbb{R}\}}$ coincide with the space of all almost-periodic functions $AP(\mathbb{R})$; see [12] for more details on the subject. Due to Theorem 2.10 (ii), we have that the translation semigroup $(T(t))_{t \geq 0}$ is frequently hypercyclic in $AP(\mathbb{R})$ and, for every $t > 0$, the operator $T(t)$ is frequently hypercyclic in $AP(\mathbb{R})$; the same holds if frequent hypercyclicity is replaced with Devaney chaoticity or topologically mixing property [18]. We can similarly prove that the translation semigroup is frequently hypercyclic in the Fréchet space $C(\mathbb{R})$ and that, for every $t > 0$, the translation operator $f \mapsto f(\cdot + t)$, $f \in C(\mathbb{R})$ is frequently hypercyclic in $C(\mathbb{R})$.

The subsequent version of Desch-Schappacher-Webb criterion for frequent hypercyclicity can be proved similarly; it is, actually, a simple consequence of Theorem 2.10 (see also [18, Theorem 3.1.36]).

THEOREM 2.12. *Let $t_0 > 0$, let $\mathbb{K} = \mathbb{C}$, and let A be a subgenerator of a global C-regularized semigroup $(S_0(t))_{t \geq 0}$ on E. Suppose that $R(C)$ is dense in E. Set $T(t)x := C^{-1}S_0(t)x$, $t \geq 0$, $x \in Z_1(A)$.*
(i) Assume that there exists an open connected subset Ω of $\mathbb{C}$, which satisfies $\sigma_p(A) \supseteq \Omega$ and intersects the imaginary axis, and $f : \Omega \to E$ is an analytic mapping satisfying $f(\lambda) \in N(A - \lambda) \setminus \{0\}$, $\lambda \in \Omega$. Assume, further, that $(x^* \circ f)(\lambda) = 0$, $\lambda \in \Omega$, for some $x^* \in E^*$, implies $x^* = 0$. Then $(S_0(t))_{t \geq 0}$ is frequently hypercyclic and each single operator $T(t_0)$ is frequently hypercyclic.

(ii) Assume that there exists an open connected subset Ω of $\mathbb{C}$, which satisfies $\sigma_p(A) \supseteq \Omega$ and intersects the imaginary axis, and $f : \Omega \to E$ is an analytic mapping satisfying $f(\lambda) \in N(A-\lambda) \setminus \{0\}$, $\lambda \in \Omega$. Put $E_0 := \mathrm{span}\{f(\lambda) : \lambda \in \Omega\}$ and $\tilde{E} := \overline{E_0}$. Then $A_{|\tilde{E}}$ is a subgenerator of a global $C_{|\tilde{E}}$-regularized semigroup $(S_0(t)_{|\tilde{E}})_{t \geq 0}$ on $\tilde{E}$, $(S_0(t)_{|\tilde{E}})_{t \geq 0}$ is frequently hypercyclic in $\tilde{E}$ and the operator $T(t_0)_{|\tilde{E}}$ is frequently hypercyclic in $\tilde{E}$.

Using Theorem 2.12 and the proof of [18, Theorem 3.1.38] (see also [19, Theorem 2.2.10]), we may deduce the following result.

THEOREM 2.13. *Let $\theta \in (0, \frac{\pi}{2})$, let $\mathbb{K} = \mathbb{C}$, and let $-A$ generate an exponentially equicontinuous, analytic strongly continuous semigroup of angle θ. Assume $n \in \mathbb{N}$, $a_n > 0$, $a_{n-i} \in \mathbb{C}$, $1 \leq i \leq n$, $D(p(A)) = D(A^n)$, $p(A) = \sum_{i=0}^n a_i A^i$ and $n(\frac{\pi}{2} - \theta) < \frac{\pi}{2}$.*
(i) Suppose there exists an open connected subset Ω of $\mathbb{C}$, satisfying $\sigma_p(-A) \supseteq \Omega$, $p(-\Omega) \cap i\mathbb{R} \neq \emptyset$, and $f : \Omega \to E$ is an analytic mapping satisfying $f(\lambda) \in N(-A - \lambda) \setminus \{0\}$, $\lambda \in \Omega$. Let $(x^* \circ f)(\lambda) = 0$, $\lambda \in \Omega$, for some $x^* \in E^*$ imply $x^* = 0$. Then, for every $\alpha \in (1, \frac{\pi}{n\pi - 2n\theta})$, there exists $\omega \in \mathbb{R}$ such that $p(A)$ generates an entire $e^{-(p(A)-\omega)^\alpha}$-regularized group $(S_0(t))_{t \in \mathbb{C}}$. Furthermore, $(S_0(t))_{t \geq 0}$ is frequently hypercyclic and, for every $t > 0$, the operator $C^{-1}S_0(t)$ is frequently hypercyclic.

(ii) Suppose there exists an open connected subset Ω of $\mathbb{C}$, satisfying $\sigma_p(-A) \supseteq \Omega$, $p(-\Omega) \cap i\mathbb{R} \neq \emptyset$, and $f : \Omega \to E$ is an analytic mapping satisfying $f(\lambda) \in N(-A - \lambda) \setminus \{0\}$, $\lambda \in \Omega$. Let E_0 and $\tilde{E}$ be as in the formulation of Theorem 2.12 (ii). Then there exists $\omega \in \mathbb{R}$ such that, for every $\alpha \in (1, \frac{\pi}{n\pi - 2n\theta})$, $p(A)$ generates an entire $e^{-(p(A)-\omega)^\alpha}$-regularized group $(S_0(t))_{t \in \mathbb{C}}$ such that $(S_0(t)_{|\tilde{E}})_{t \geq 0}$ is frequently hypercyclic and, for every $t > 0$, the operator $C^{-1}S_0(t)_{\tilde{E}}$ is frequently hypercyclic.

Theorem 2.10, Theorem 2.12 and Theorem 2.13 can be applied in a great number of concrete situations. In what follows, we will continue our analyses from [18, Example 3.1.40, Example 3.1.41, Example 3.1.44].

EXAMPLE 2.14. (i) ([11]) Consider the following convection-diffusion type equation of the form
$$\begin{cases} u_t = au_{xx} + bu_x + cu := -Au, \\ u(0,t) = 0, \ t \geq 0, \\ u(x,0) = u_0(x), \ x \geq 0. \end{cases}$$

As it is well known, the operator $-A$, acting with domain $D(-A) = \{f \in W^{2,2}([0,\infty)) : f(0) = 0\}$, generates an analytic strongly continuous semigroup of angle $\pi/2$ in the space $E = L^2([0,\infty))$, provided $a, b, c > 0$ and $c < \frac{b^2}{2a} < 1$. The same conclusion holds true if we consider the operator $-A$ with the domain $D(-A) = \{f \in W^{2,1}([0,\infty)) : f(0) = 0\}$ in $E = L^1([0,\infty))$. Set
$$\Omega := \left\{\lambda \in \mathbb{C} : \left|\lambda - \left(c - \frac{b^2}{4a}\right)\right| \leq \frac{b^2}{4a}, \ \Im\lambda \neq 0 \text{ if } \Re\lambda \leq c - \frac{b^2}{4a}\right\}.$$

Let $p(x) = \sum_{i=0}^n a_i x^i$ be a nonconstant polynomial such that $a_n > 0$ and $p(-\Omega) \cap i\mathbb{R} \neq \emptyset$ (this condition holds provided that $a_0 \in i\mathbb{R}$). An application of Theorem 2.13 (i)

shows that there exists an injective operator $C \in L(E)$ such that $p(A)$ generates an entire C-regularized group $(S_0(t))_{t\geq 0}$ satisfying that $(S_0(t))_{t\geq 0}$ is frequently hypercyclic and each single operator $T(t_0)$ is frequently hypercyclic ($t_0 > 0$).

(ii) ([17]) Let X be a symmetric space of non-compact type (of rank one) and $p > 2$. Then there exists an injective operator $C \in L(L_\natural^p(X))$ such that for each $c \in \mathbb{R}$ the operator $\Delta_{X,p}^\natural - c$ generates an entire C-regularized group $(S_0(t))_{t\geq 0}$ in $L_\natural^p(X)$. Furthermore, owing to [17, Theorem 3.1] and Theorem 2.13 (i), there exists a number $c_p > 0$ such that, for every $c > c_p$, the semigroup $(S_0(t))_{t\geq 0}$ is frequently hypercyclic in $L_\natural^p(X)$ and each single operator $T(t_0)$ is frequently hypercyclic in $L_\natural^p(X)$ ($t_0 > 0$).

(iii) ([19]) Suppose that $\alpha > 0$, $\tau \in i\mathbb{R} \setminus \{0\}$ and $E := BUC(\mathbb{R})$. After the usual matrix conversion to a first order system, the equation $\tau u_{tt} + u_t = \alpha u_{xx}$ becomes

$$\frac{d}{dt}\vec{u}(t) = P(D)\vec{u}(t), \ t \geq 0, \text{ where } D \equiv -i\frac{d}{dx}, \ P(x) \equiv \begin{bmatrix} 0 & 1 \\ -\frac{\alpha}{\tau}x^2 & -\frac{1}{\tau} \end{bmatrix},$$

and $P(D)$ acts on $E \oplus E$ with its maximal distributional domain. The polynomial matrix $P(x)$ is not Petrovskii correct and applying [?, Theorem 14.1] we get that there exists an injective operator $C \in L(E \oplus E)$ such that $P(D)$ generates an entire C-regularized group $(S_0(t))_{t\geq 0}$, with $R(C)$ dense. Define the numbers ω_1, $\omega_2 \in [0, +\infty]$ and functions $\psi_{r,j} \in E \oplus E$ ($r \in \mathbb{R}$, $j = 1, 2$) as it has been done in [18, Example 3.1.44]; $\tilde{E} := \overline{\text{span}}\{\psi_{r,j} : r \in \mathbb{R}, \ j = 1, 2\}$. Due to Theorem 2.10 (ii), we have that $(S_0(t)_{|\tilde{E}})_{t\geq 0}$ is frequently hypercyclic in $\tilde{E}$ and, for every $t > 0$, the operator $C^{-1}S_0(t)_{|\tilde{E}}$ is frequently hypercyclic in $\tilde{E}$.

(iv) ([10]) Denote by $(W_Q(t))_{t\geq 0}$ the $e^{-(-B^2)^N}$-regularized semigroup generated by the operator $Q(B)$, whose existence has been proved in [10, Lemma 5.2]. If the requirement stated in the formulation of [10, Theorem 5.3] holds, then $(W_Q(t))_{t\geq 0}$ and each single operator $e^{(-B^2)^N}W_Q(t_0)$ is frequently hypercyclic ($t_0 > 0$); this simply follows from an application of Theorem 2.12 (i).

(v) ([18]) Finally, we turn our attention to integrated semigroups. Let $n \in \mathbb{N}$, $\rho(t) := \frac{1}{t^{2n}+1}$, $t \in \mathbb{R}$, $Af := f'$, $D(A) := \{f \in C_{0,\rho}(\mathbb{R}) : f' \in C_{0,\rho}(\mathbb{R})\}$, $E_n := (C_{0,\rho}(\mathbb{R}))^{n+1}$, $D(A_n) := D(A)^{n+1}$ and $A_n(f_1, \cdots, f_{n+1}) := (Af_1 + Af_2, Af_2 + Af_3, \cdots, Af_n + Af_{n+1}, Af_{n+1})$, $(f_1, \cdots, f_{n+1}) \in D(A_n)$. Then $\pm A_n$ generate global polynomially bounded n-times integrated semigroups $(S_{n,\pm}(t))_{t\geq 0}$ and neither A_n nor $-A_n$ generates a local $(n-1)$-times integrated semigroup. If we denote by $G_{\pm,n}$ the associated distribution semigroups generated by $\pm A_n$, then for every $\varphi_1, \cdots, \varphi_{n+1} \in \mathcal{D}$, we have:

$$G_{\pm,n}(\delta_t)(\varphi_1, \cdots, \varphi_{n+1})^T = (\psi_1, \cdots, \psi_{n+1})^T,$$

where
$$\psi_i(\cdot) = \sum_{j=0}^{n+1-i} \frac{(\pm t)^j}{j!}\varphi_{i+j}^{(j)}(\cdot \pm t), \quad 1 \leq i \leq n+1.$$

Set $E_0 := \mathcal{D}^{n+1}$ and $S_k(\varphi_1, \cdots, \varphi_{n+1})^T := (\phi_1, \cdots, \phi_{n+1})^T$, where

$$\phi_i(\cdot) = \sum_{j=0}^{n+1-i} \frac{(\mp k t_0)^j}{j!} \varphi_{i+j}^{(j)}(\cdot \mp k t_0), \quad 1 \le i \le n+1,$$

for any $k \in \mathbb{N}$, $t_0 > 0$ and $\varphi_1, \cdots, \varphi_{n+1} \in \mathcal{D}$. Then we can simply verify (see also [18, Example 3.2.39]) that the conditions of Theorem 2.5 hold with $C = (\lambda \mp A_n)^{-n}$, where $\rho(\pm A_n) \ni \lambda > 0$ is sufficiently large, since the series in (i)-(iii) from the formulation of this theorem converge absolutely. Hence, the integrated semigroups $(S_{n,\pm}(t))_{t \ge 0}$ are frequently hypercyclic in E_n and for each each number $t_0 > 0$ the single operators $G_{\pm,n}(\delta_{t_0})$ are frequently hypercyclic in E_n.

It is worth noting that, for any C-regularized semigroup or integrated semigroup considered above, say $(S(t))_{t \ge 0}$, any finite direct sum $(S(t) \oplus S(t) \oplus \cdots \oplus S(t))_{t \ge 0}$ is again frequently hypercyclic or subspace frequently hypercyclic, with the meaning clear. The same holds for finite direct sums of considered single operators (cf. [7] for more details about this topic).

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