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LOCATION AND WEIGHT DISTRIBUTION OF KEY ERRORS

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Abstract. In this presentation, we give necessary and sufficient conditions (lower and upper bounds) for the existence of linear codes capable of identifying the portion of the codeword which is corrupted by errors named as *key errors*. An example of such a code is provided. Comparisons among the number of parity check digits of linear codes detecting/locating/correcting key errors are provided. A result on minimum weight of key errors in Hamming sense is also included in the paper.

1. Preliminaries

Error control coding scheme is mainly used for distant communication to protect the information which may get corrupted during the process of communication. But with the advancement of information technology, it is quite possible that the information may be disturbed at entry level also. There are communication channels like automata or electronics devices where entry level error occurs.

Consider the example of keyboard of a computer which has keys for various alphabets, numerical and symbols. When one types a number or a symbol, there is always a possibility to make mistake in typing. One can make a mistake by pressing the wrong key on either side of the correct key. We may call such errors *key errors*. Such errors are already studied by Sharma and Gaur in [9] with respect to S-K metric. Detection and correction of key errors are studied in [1, 2] and the key errors are defined as follows.

Definition 1.1. An i-key error of length b ($i = 1, 2, \dots n$) is an n-tuple such that the i^{th} component is non-zero and all other non-zero components are confined up to b consecutive positions (if they exist) preceding or succeeding the i^{th} component.

Note that the entry error (i^{th} component) of such error could be at any position from 1^{st} to n^{th} position. Suppose that the 1^{st} position contains the entry error; then

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possibility of occurring errors are on the b consecutive positions to the right side of the 1^{st} position. If the entry error is in the 2^{nd} position, then other errors may be in one position left of the 2^{nd} position and the b consecutive positions, immediately right of the 2^{nd} position. If the entry error is in the 3^{rd} position, the two immediate positions left of the 3^{rd} position and immediate b consecutive positions to the right of the 3^{rd} position contain other errors. Continuing in the same way, if the entry error occurs in the last position, i.e., the n^{th} position, then the other errors will be confined to the b consecutive positions immediately left of the n^{th} position.

Just like in [1,2], we can consider examples of key errors of length 2 in a vector of length 6 over $GF(3)$ as follows:

$$(0\ \underbrace{21}_{2}\ \underbrace{2}_{\text{entry error}}\ \underbrace{12}_{2}),\quad (0\ \underbrace{21}_{2}\ \underbrace{1}_{\text{entry error}}\ \underbrace{02}_{2}),\quad (0\ \underbrace{12}_{2}\ \underbrace{2}_{\text{entry error}}\ \underbrace{00}_{2}),$$

$$(000\ \underbrace{11}_{2}\ \underbrace{1}_{\text{entry error}}),\quad (\underbrace{11}_{2}\ \underbrace{1}_{\text{entry error}}\ \underbrace{22}_{2}\ 0),\quad (\underbrace{1}_{1}\ \underbrace{1}_{\text{entry error}}\ \underbrace{12}_{2}\ 00),\quad \text{etc.}$$

In this paper, we have studied linear code that will locate key errors. Such codes are called ‘Error Locating Codes’. The concept of ‘Error Locating Codes’, a middle concept between error detection and error correction, was first proposed by Wolf and Elspas [10] in 1963. They subdivided the code into some mutually exclusive sub-blocks. While decoding, the error occurring in a sub-block can be detected and in addition, which particular sub-block contains the error can also be identified.

The efficiency of detection/location/correction of error by a code can be improved by minimizing the number of parity check digits of the code. Although it is not always possible to find the required exact number of such digits, but it is possible to obtain the bound on the number of such digits and this was initiated by Hamming [3] where also a technique was given for construction of such codes.

Let us consider an (n, k) linear code over $GF(q)$ that is divided into m mutually exclusive sub-blocks, each of length t and let H be the parity check matrix of the code. The $(n = mt, k)$ code capable of locating key errors of length at most b occurring within a sub-block of length t is called *Single Blockwise Key Error Locating* code, or $SBK_{b/t}EL$ code. To be an $SBK_{b/t}EL$ code, the following two conditions must be satisfied:

(I) $eH^T \neq 0$ where e is any key error of length at most b within a sub-block.

(II) $e_iH^T \neq e_jH^T$, where $1 \leq i, j \leq m,\ i \neq j$, and e_i and e_j are any key errors of length at most b occurring in the i^{th} and j^{th} sub-blocks.

In [1,2], the author derives lower and upper bounds on the number of parity check for *key error of length up to b detecting* (K_bED) code and *key error of length up to b correcting* (K_bEC) code respectively. The present paper derives such bounds for an $SBK_{b/t}EL$ code. We also provide comparison among the numbers of parity check digits of these three types of codes. Further, we obtain a combinatorial result on weight of key errors analogous to the famous Plotkin bound [6]. The weight of a vector is considered in Hamming sense as the number of non-zero entries.

The paper is organized as follows. The contents of Section 2 are derivation of lower and upper bounds on the number of parity check digits of an $SBK_{b/t}EL$ code along with an example. In Section 3, Plotkin type bound is presented. Finally, Section 4 gives comparison of parity check digits among K_bED, K_bEC and $SBK_{b/t}EL$ codes.

2. Location of key errors

Firstly, we derive a lower bound on the number of parity check digits needed for an $SBK_{b/t}EL$ code. In order to do it, we apply the technique of Peterson and Weldon [5, Theorem 4.16].

Theorem 2.1. *For any $(n = mt, k)$ $SBK_{b/t}EL$ code $(t > 2b)$, the number of parity check digits r $(= n - k)$ satisfies*

$$q^r \geq \begin{cases} 1 + mb, & \text{for } q = 2, \\ 1 + mb(q-1)(q-2), & \text{for } q \neq 2. \end{cases}$$

Proof. Since the (n, k) linear code over $GF(q)$ detects key errors of length at most b within a sub-block and identifies the corrupted sub-block, the number of distinct syndromes according to conditions (I) and (II) has to be less than or equal of q^r, maximum possible number of distinct syndromes.

Let X be the set consisting of n-tuples such that the non-zero components are confined to the first $2b$ positions in any one sub-block (as considered in [1]) as follows.

(i) For $q = 2$, the first $2b$ positions are

$$(\overbrace{x00\ldots00}^{b}\ \overbrace{y00\ldots00}^{b}),$$
$$(\overbrace{0x0\ldots00}^{b}\ \overbrace{0y0\ldots00}^{b}),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$(\overbrace{000\ldots0x}^{b}\ \overbrace{000\ldots0y}^{b}),$$

where $x = y = 1$

(ii) For $q \neq 2$, the first $2b$ positions are

$$(\overbrace{x00\ldots00}^{b}\ \overbrace{y00\ldots00}^{b}),$$
$$(\overbrace{0x0\ldots00}^{b}\ \overbrace{0y0\ldots00}^{b}),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$(\overbrace{000\ldots0x}^{b}\ \overbrace{000\ldots0y}^{b}),$$

where $x, y \in GF(q) \setminus \{0\}$ and $x \neq y$.

Then, we have that the syndromes of all elements of X are all distinct (see [1]). Also by condition (II), the syndromes of vectors which are key errors of length at most b, whether in the same sub-block or in different sub-blocks, must be distinct.

By [1], the number of elements of X, excluding the vector of all zeros, is

(i) b for $q = 2$, (ii) $b(q-1)(q-2)$ for $q \neq 2$.

The above number of distinct non-zero syndromes is counted corresponding to vectors lying within a single sub-block. As there are m sub-blocks in all, the number of distinct non-zero syndromes, including the all zero vectors, is at least

(i) $1 + mb$ for $q = 2$, (ii) $1 + mb(q-1)(q-2)$ for $q \neq 2$.

Therefore, we must have

$$q^r \geq \begin{cases} 1 + mb, & \text{for } q = 2, \\ 1 + mb(q-1)(q-2), & \text{for } q \neq 2. \end{cases} \qquad \square$$

An *upper* bound on the number of check digits required for the construction of an $SBK_{b/t}EL$ code is given in the next theorem where the technique of Varshomov-Gilbert-Sacks bound (refer to Sacks [7] and Peterson and Weldon [5, Theorem 4.7]) is followed. The bound establishes the existence of such codes.

THEOREM 2.2. *For the existence of an* $(n = mt, k)$ $SBK_{b/t}EL$ *code* $(t > 4b + 2)$, *the minimum number of parity check digits* r $(= n - k)$ *required is at least*

$$1 + \log_q \left[\left(1 + q^{b-1}(q-1)q^b\right) \times \left(1 + (m-1)\left\{ \frac{q^{2b+1} - q}{q+1} \right.\right.\right.$$
$$\left.\left.\left. + (t - 2b)(q-1)\left(\frac{q^{2b+1} - q}{q+1} + 1\right) + \frac{q^{2b+1} - q^3 + (q^2 - 1)(b + q)}{(q+1)^2} \right\}\right)\right].$$

Proof. The existence can be established if we can always construct an $(n - k) \times n$ parity check matrix H for such a code. In order to construct parity check matrix H, we follow the technique as below.

We first assume that the columns of the first $m - 1$ sub-blocks of H and the first $j - 1$ columns $h_1, h_2, \ldots, h_{j-1}$ of the m^{th} sub-block of H have been appropriately added. We now add j^{th} column h_j of the m^{th} sub-block of the matrix H as follows.

According to condition (I) for being nonzero syndromes of key error of length at most b within a sub-block, h_j should not be a linear combination of immediately preceding consecutive $2b$ columns such that the coefficient of the preceding $(b+1)^{th}$ column is non-zero. In other words,

$$\begin{aligned} h_j \neq\ & (u_1 h_{j-1} + u_2 h_{j-2} + \cdots + u_{b-1} h_{j-b+1}) + u_b h_{j-b} \\ & + \ (u_{b+1} h_{j-b-1} + u_{b+2} h_{j-b-2} + \cdots + u_{2b} h_{j-2b}), \end{aligned} \tag{1}$$

where $u_i \in GF(q)$ and $u_b \neq 0$. From [1], we know the number of the coefficients u_i satisfying $u_i \in GF(q)$ and $u_b \neq 0$, including the zero vector in (1), and this number is

$$1 + q^{b-1}(q-1)q^b. \tag{2}$$

Further, according to condition (II), the syndromes of such key errors need to be distinct in different sub-blocks. Therefore, h_j can be added provided that

$$\begin{aligned} h_j \neq\ & (u_1 h_{j-1} + u_2 h_{j-2} + \cdots + u_{b-1} h_{j-b+1}) + u_b h_{j-b} \\ & + (u_{b+1} h_{j-b-1} + u_{b+2} h_{j-b-2} + \cdots + u_{2b} h_{j-2b}) \\ & + (v_{l+1} h_{l+1} + v_{l+2} h_{l+2} + \cdots + v_{l+(2b+1)} h_{l+(2b+1)}), \end{aligned} \tag{3}$$

where $u_i, v_i \in GF(q)$, $u_b \neq 0$ and h_{l+i}'s in the last bracket form a pattern of key errors of length at most b in any one sub-block among the previous $m - 1$ sub-blocks.

The number of different ways the coefficients u_i on R.H.S. of (3) can be selected is $1 + q^{b-1}(q-1)q^b$. Enumeration of the coefficients v_i's is equivalent to the enumeration of the number of key errors of length at most b in a vector of length t. From [2], we can obtain this number (excluding the vector of all zeros) to be

$$\frac{q^{2b+1}-q}{q+1}+(t-2b)(q-1)\left(\frac{q^{2b+1}-q}{q+1}+1\right)+\frac{q^{2b+1}-q^3}{(q+1)^2}+\frac{(q-1)(b+q)}{q+1}.$$

Since there are $m-1$ previously chosen sub-blocks, therefore the number of linear combinations of v_i's on R.H.S. of (3) becomes

$$(m-1)\left\{\frac{q^{2b+1}-q}{q+1}+(t-2b)(q-1)\left(\frac{q^{2b+1}-q}{q+1}+1\right)+\frac{q^{2b+1}-q^3}{(q+1)^2}+\frac{(q-1)(b+q)}{q+1}\right\}.$$

So, for location, the number of linear combinations that h_j cannot take is the product

$$\left\{1+q^{b-1}(q-1)q^b\right\}\times$$
$$(m-1)\left\{\frac{q^{2b+1}-q}{q+1}+(t-2b)(q-1)\left(\frac{q^{2b+1}-q}{q+1}+1\right)+\frac{q^{2b+1}-q^3}{(q+1)^2}+\frac{(q-1)(b+q)}{q+1}\right\}. \quad (4)$$

Therefore, for detecting and locating of key errors of length at most b, the total number of linear combinations which h_j cannot take is the sum of the quantities computed in (2) and (4).

Thus, taking the worst situation when all these combinations yield distinct sums, the j^{th} column h_j to the m^{th} sub-block can be added H provided that

$$q^{n-k}>\left[1+q^{b-1}(q-1)q^b\right]\times \quad (5)$$
$$\left[1+(m-1)\left\{\frac{q^{2b+1}-q}{q+1}+(t-2b)(q-1)\left(\frac{q^{2b+1}-q}{q+1}+1\right)+\frac{q^{2b+1}-q^3+(q^2-1)(b+q)}{(q+1)^2}\right\}\right].$$

This completes the proof of the theorem. □

Remark 2.3. For $m=1$, the inequality (5) coincides with the one in [1, Theorem 2]. The inequality (5) depends on the value of t (which may be chosen to be smaller quantity), whereas the R.H.S. of the inequality of [2, Theorem 2.2] depends on n. As a result, the quantity on R.H.S. of [2, Theorem 2.2] gets bigger than that of the inequality (5).

Example 2.4. Consider the 9×12 parity check matrix H of a $(12,3)$ linear code over $GF(2)$. The matrix H is constructed following the technique used in the proof of Theorem 2.2 by taking $q=2$, $b=2$, $t=6$ and $m=2$.

$$H=\begin{bmatrix}
1&0&0&0&0&1&0&0&0&1&0&0\\
1&1&0&0&0&1&0&0&0&1&1&0\\
0&1&1&0&0&0&0&0&0&1&1&1\\
0&0&1&1&0&0&0&0&0&1&1&0\\
0&0&0&1&1&0&0&0&0&1&1&1\\
0&0&0&0&1&0&1&0&0&1&1&0\\
0&0&0&0&0&0&1&1&0&1&1&1\\
0&0&0&0&0&0&0&1&1&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0&1
\end{bmatrix}$$

Table 1: Error Pattern-Syndrome

Error Patterns	Syndromes	Error Patterns	Syndromes
100000 000000	110000000	000000 100000	000001100
110000 000000	101000000	000000 110000	000001010
101000 000000	111100000	000000 101000	000001111
111000 000000	100100000	000000 111000	000001001
010000 000000	011000000	000000 010000	000000110
011000 000000	010100000	000000 011000	000000101
010100 000000	011110000	000000 010100	111111010
011100 000000	010010000	000000 011100	111111001
110100 000000	101110000	000000 110100	111110110
111100 000000	100010000	000000 111100	111110101
001000 000000	001100000	000000 001000	000000011
001100 000000	001010000	000000 001100	111111111
001010 000000	001111000	000000 001010	011111101
001110 000000	001001000	000000 001110	100000001
101100 000000	111010000	000000 101100	111110011
101010 000000	111111000	000000 101010	011110001
101110 000000	111001000	000000 101110	100001101
011010 000000	010111000	000000 111110	100001011
011110 000000	010001000	000000 011010	011111011
111010 000000	100111000	000000 111010	011110111
111110 000000	100001000	000000 111110	100001011
000100 000000	000110000	000000 000100	111111100
000110 000000	000101000	000000 000110	100000010
000101 000000	110110000	000000 000101	110101001
000111 000000	110101000	000000 000111	101010111
010110 000000	011101000	000000 010110	100000100
010101 000000	101110000	000000 010101	110101111
010111 000000	101101000	000000 010111	101010001
001101 000000	111010000	000000 001101	110101010
001111 000000	111001000	000000 001111	101010100
011101 000000	100010000	000000 011101	110101100
011111 000000	100001000	000000 011111	101010010
000010 000000	000011000	000000 000010	011111110
000011 000000	110011000	000000 000011	010101011
001011 000000	111111000	000000 001011	010101000
000001 000000	110000000	000000 000001	001010101

From Table 1 given above, we get all non-zero and distinct syndromes in different sub-blocks arising out of any key error of length at most 2. Therefore, any key error of length at most 2 can be located by the code of a null space of H. This example is created by modifying [2, Example 2.1] so as to make the conditions (I) and (II) satisfied.

3. Plotkin type bound

In coding theory, Plotkin bound is a well known and important bound which deals with the minimum weight in a group of vectors. For results on weight of vectors, one may refer to [4, 8]. The following theorem (which is equivalent to Plotkin bound [6], also Peterson and Weldon [5, Theorem 4.1]) is a result in that direction.

Theorem 3.1. *The minimum weight of a vector having all key errors of length at most b in the linear space of all n-tuples is at most*

$$\frac{A}{\frac{q^{2b+1}-q}{q+1}+(n-2b)(q-1)\left(\frac{q^{2b+1}-q}{q+1}+1\right)+\frac{q^{2b+1}-q^3}{(q+1)^2}+\frac{(q-1)(b+q)}{q+1}},$$

where

$$\begin{aligned} A = &\sum_{j=2}^{b}\sum_{l=j}^{2j-2}\sum_{i=0}^{l}(i+2)\binom{l}{i}(q-1)^{2+i}+\sum_{l=1}^{b}\sum_{i=0}^{l}(i+1)\binom{l}{i}(q-1)^{1+i}\\ &+(n-2b)\left\{\sum_{l=1}^{b}\sum_{i=0}^{2l-1}(2+i)\binom{2l-1}{i}(q-1)^{2+i}+(q-1)\right\}\\ &+\sum_{l=2}^{b}(b-l+1)\sum_{i=0}^{2l-3}(2+i)\binom{2l-3}{i}(q-1)^{2+i}+b(q-1). \end{aligned}$$

Proof. The total weight of key errors of length at most b, when we take the entry error of the key errors from 1^{st} to b^{th} position is given by

$$\begin{aligned} &\sum_{i=0}^{b}(i+1)(q-1)^{i+1}\binom{b}{i}\\ &+\left\{\sum_{i=0}^{b}(i+2)\binom{b}{i}(q-1)^{2+i}+\sum_{i=0}^{b-1}(i+1)\binom{b-1}{i}(q-1)^{i+1}\right\}\\ &+\left\{\sum_{i=0}^{b+1}(i+2)\binom{b+1}{i}(q-1)^{2+i}+\sum_{i=0}^{b-1}(i+2)\binom{b-1}{i}(q-1)^{2+i}+\sum_{i=0}^{b-2}(i+1)\binom{b-2}{i}(q-1)^{1+i}\right\}\\ &+\left\{\sum_{i=0}^{b+2}(i+2)\binom{b+2}{i}(q-1)^{2+i}+\sum_{i=0}^{b}(i+2)\binom{b}{i}(q-1)^{2+i}+\sum_{i=0}^{b-2}(i+2)\binom{b-2}{i}(q-1)^{2+i}\right.\\ &\quad\left.+\sum_{i=0}^{b-3}(i+1)\binom{b-3}{i}(q-1)^{1+i}\right\}\\ &+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

$$+\Bigg\{\sum_{i=0}^{2b-2}(i+2)\binom{2b-2}{i}(q-1)^{2+i}+\sum_{i=0}^{2b-4}(i+2)\binom{2b-4}{i}(q-1)^{2+i}$$

$$+\sum_{i=0}^{2b-6}\binom{2b-6}{i}(q-1)^{2+i}+\ldots+\sum_{i=0}^{2}(i+2)\binom{2}{i}(q-1)^{2+i}+\sum_{i=0}^{1}(1+i)\binom{1}{i}(q-1)^{1+i}\Bigg\},$$

which is equal to

$$\sum_{l=b}^{2b-2}\sum_{i=0}^{l}(i+2)\binom{l}{i}(q-1)^{2+i}+\sum_{l=b-1}^{2b-4}\sum_{i=0}^{l}(i+2)\binom{l}{i}(q-1)^{2+i}+\ldots$$

$$+\sum_{l=2}^{2}\sum_{i=0}^{l}(i+2)\binom{l}{i}(q-1)^{2+i}+\sum_{l=1}^{b}\sum_{i=0}^{l}(i+1)\binom{l}{i}(q-1)^{1+i}$$

$$=\sum_{j=2}^{b}\sum_{l=j}^{2j-2}\sum_{i=0}^{l}(i+2)\binom{l}{i}(q-1)^{2+i}+\sum_{l=1}^{b}\sum_{i=0}^{l}(i+1)\binom{l}{i}(q-1)^{1+i}.$$

Taking the entry error from $(b+1)^{th}$ position to $(n-b)^{th}$ position, we get the total weight of key errors of at most length b as

$$(n-2b)\Bigg\{\sum_{i=0}^{2b-1}(2+i)\binom{2b-1}{i}(q-1)^{2+i}+\sum_{i=0}^{2b-3}(2+i)\binom{2b-3}{i}(q-1)^{2+i}$$

$$+\sum_{i=0}^{2b-5}(2+i)\binom{2b-5}{i}(q-1)^{2+i}+\cdots+\sum_{i=0}^{1}(2+i)\binom{1}{i}(q-1)^{2+i}+(q-1)\Bigg\}$$

$$=(n-2b)\Bigg\{\sum_{l=1}^{b}\sum_{i=0}^{2l-1}(2+i)\binom{2l-1}{i}(q-1)^{2+i}+(q-1)\Bigg\}.$$

If the entry error of key errors is in the last b positions, the total weight of key errors of length at most b is

$$\Bigg\{\sum_{i=0}^{2b-3}(2+i)\binom{2b-3}{i}(q-1)^{2+i}+\sum_{i=0}^{2b-5}(2+i)\binom{2b-5}{i}(q-1)^{2+i}+\ldots$$

$$+\sum_{i=0}^{3}(2+i)\binom{3}{i}(q-1)^{2+i}+\sum_{i=0}^{1}(2+i)\binom{1}{i}(q-1)^{2+i}+(q-1)\Bigg\}$$

$$+\Bigg\{\sum_{i=0}^{2b-5}(2+i)\binom{2b-5}{i}(q-1)^{2+i}+\sum_{i=0}^{2b-7}(2+i)\binom{2b-7}{i}(q-1)^{2+i}+\ldots$$

$$+\sum_{i=0}^{3}(2+i)\binom{3}{i}(q-1)^{2+i}+\sum_{i=0}^{1}(2+i)\binom{1}{i}(q-1)^{2+i}+(q-1)\Bigg\}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\left\{\sum_{i=0}^{5}(2+i)\binom{5}{i}(q-1)^{2+i}+\sum_{i=0}^{3}(2+i)\binom{3}{i}(q-1)^{2+i}+\sum_{i=0}^{1}(2+i)\binom{1}{i}(q-1)^{2+i}+(q-1)\right\}$$
$$+\left\{\sum_{i=0}^{3}(2+i)\binom{3}{i}(q-1)^{2+i}+\sum_{i=0}^{1}(2+i)\binom{1}{i}(q-1)^{2+i}+(q-1)\right\}$$
$$+\left\{+\sum_{i=0}^{1}(2+i)\binom{1}{i}(q-1)^{2+i}+(q-1)\right\}+(q-1),$$

which is equal to

$$\sum_{l=2}^{b}(b-l+1)\sum_{i=0}^{2l-3}(2+i)\binom{2l-3}{i}(q-1)^{2+i}+b(q-1).$$

Hence, the total weight of key errors of length at most b is

$$\sum_{j=2}^{b}\sum_{l=j}^{2j-2}\sum_{i=0}^{l}(i+2)\binom{l}{i}(q-1)^{2+i}+\sum_{l=1}^{b}\sum_{i=0}^{l}(i+1)\binom{l}{i}(q-1)^{1+i}$$
$$+(n-2b)\left\{\sum_{l=1}^{b}\sum_{i=0}^{2l-1}(2+i)\binom{2l-1}{i}(q-1)^{2+i}+(q-1)\right\}$$
$$+\sum_{l=2}^{b}(b-l+1)\sum_{i=0}^{2l-3}(2+i)\binom{2l-3}{i}(q-1)^{2+i}+b(q-1)=A\ (say).$$

Further, the total number of key errors of length at most b is given by [2] and it is

$$\frac{q^{2b+1}-q}{q+1}+(n-2b)(q-1)\left(\frac{q^{2b+1}-q}{q+1}+1\right)+\frac{q^{2b+1}-q^3}{(q+1)^2}+\frac{(q-1)(b+q)}{q+1}.$$

Since the minimum weight of a vector in the linear space of all n-tuples can be at most equal to the average weight, so an upper bound on minimum weight of key errors of length at most b is the average weight, which is

$$\frac{A}{\frac{q^{2b+1}-q}{q+1}+(n-2b)(q-1)\left(\frac{q^{2b+1}-q}{q+1}+1\right)+\frac{q^{2b+1}-q^3}{(q+1)^2}+\frac{(q-1)(b+q)}{q+1}}.$$

This completes the theorem. □

4. Comparison

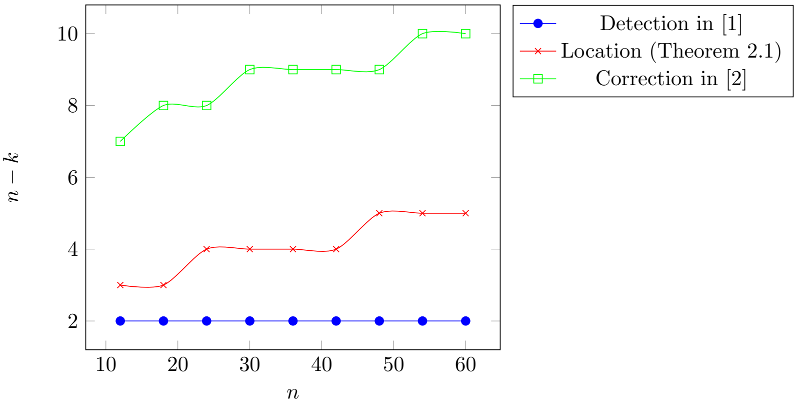
Detection of errors, location of errors and correction of errors are all important in connection to a system or situation. Accordingly, codes are constructed to deal with these problems. The numbers of parity check digits required for the three types of codes are different. The lesser number of parity check digits improves the rate of information.

In this section, we make comparisons between the necessary (lower bound) and sufficient numbers (upper bound) of parity check digits required for $SBK_{b/t}EL$ codes (Theorem 2.1 and Theorem 2.2) with the K_bED codes that detect all key errors of length at most b [1, Theorem 1 and Theorem 2], and with the K_bEC codes that correct all key errors of length at most b [2, Theorem 2.1 and Theorem 2.2].

First, we give a comparison between the necessary number of check digits needed for a K_bED code [1, Theorem 1], K_bEC code [2, Theorem 2.1] and our $SBK_{b/t}EL$ code (Theorem 2.1).

Table and Figure 1: Comparison of necessary number of check digits for codes detecting, correcting & locating key errors

m	t	b	n	$n-k$	$n-k$	$n-k$
				for detection in [1]	for location in Theorem 2.1	for correction in [2]
2	6	2	12	2	3	7
3	6	2	18	2	3	8
4	6	2	24	2	4	8
5	6	2	30	2	4	9
6	6	2	36	2	4	9
7	6	2	42	2	4	9
8	6	2	48	2	5	9
9	6	2	54	2	5	10
10	6	2	60	2	5	10

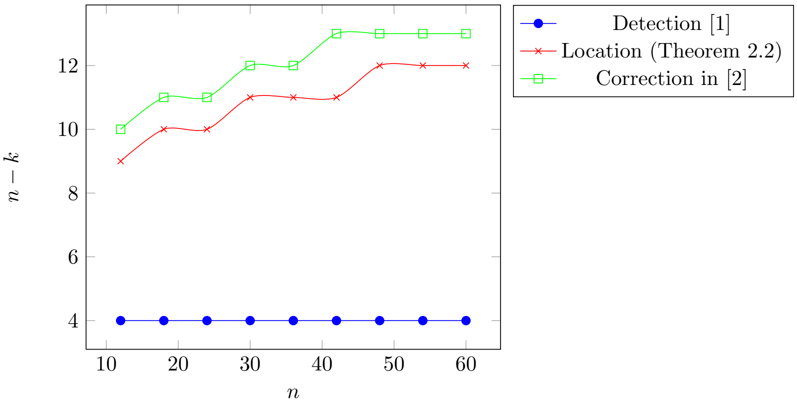


It is evident from Table and Figure 1 that the necessary number of parity check digits needed for an $SBK_{b/t}EL$ code lies between the necessary number of parity check digits needed for a K_bED code and K_bEC code. The necessary number of parity check digits for K_bED, $SBK_{b/t}EL$ and K_bEC codes are in increasing order.

A similar comparison between the sufficient number of parity check digit required for the existence of a K_bED code [1, Theorem 2], K_bEC code [2, Theorem 2.2] and our $SBK_{b/t}EL$ code (Theorem 2.2) is given in the following Table and Figure 2. Here also, the sufficient number of parity check digits for K_bED, $SBK_{b/t}EL$ and K_bEC codes are in ascending order.

Table and Figure 2: Comparison of sufficient number of check digits for codes detecting, correcting & locating key errors

m	t	b	n	$n-k$ for detection in [1]	$n-k$ for location in Theorem 2.1	$n-k$ for correction in [2]
2	6	2	12	4	9	10
3	6	2	18	4	10	11
4	6	2	24	4	10	11
5	6	2	30	4	11	12
6	6	2	36	4	11	12
7	6	2	42	4	11	13
8	6	2	48	4	12	13
9	6	2	54	4	12	13
10	6	2	60	4	12	13



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