MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
74, 2 (2022), 130–140
June 2022

research paper
оригинални научни рад

FIXED POINTS OF ALMOST SUZUKI TYPE $\mathcal{Z}_s$-CONTRACTIONS IN S-METRIC SPACES

G. V. R. Babu, P. Durga Sailaja and G. Srichandana

Abstract. In this paper, we introduce almost Suzuki type $\mathcal{Z}_s$-contractions and prove the existence and uniqueness of fixed points of such mappings in complete S-metric spaces. Our results generalize Theorem 1 from [N. Mlaiki, N. Yılmaz Özgür, Nihal Taş, Mathematics, 7 (583) 2019, 12 pages] and Theorem 3.1 from [S. Sedghi, N. Shobe, A. Aliouche, Mat. Vesnik, 64 (3) (2012), 258-266]. We give illustrative examples in support of our result.

1. Introduction and preliminaries

In 2008, Suzuki [19] defined a new generalized Banach contraction and proved the existence and uniqueness of fixed points for this contraction in compact metric spaces. After this several authors have extended and generalized the result of Suzuki [19] in different directions [1, 5, 13]. In 2015, Khojasteh, Shukla and Radenović [12] introduced simulation functions and $\mathcal{Z}$-contractions which generalize the Banach contraction. Following this domain of research, many authors introduced $\mathcal{Z}$-contractions involving simulation functions and proved fixed point results on various types of metric spaces. For more works on this, we refer to [2, 4, 7, 11, 16, 20].

Definition 1.1 ([12]). A mapping $\zeta : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ is called a simulation function if it satisfies the following conditions:
(ζ_1) $\zeta(0, 0) = 0$;

(ζ_2) $\zeta(t, s) < s - t$ for all $t, s > 0$;

(ζ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, +\infty)$ such that $\lim\limits_{n\to+\infty} t_n = \lim\limits_{n\to+\infty} s_n > 0$, then $\limsup\limits_{n\to+\infty} \zeta(t_n, s_n) < 0$.

2020 Mathematics Subject Classification: 47H10, 54H25.
Keywords and phrases: S-metric space; $\mathcal{Z}$-contraction; simulation function; $\mathcal{Z}_s$-contraction; almost Suzuki type $\mathcal{Z}_s$-contraction.

We denote the set of all simulation functions by $\mathcal{Z}$. The following are examples of simulation functions.

Example 1.2 ([4,12]). Let $\zeta : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ be defined by:
(i) $\zeta(t,s) = \lambda s - t$ for all $t,s \in [0,+\infty)$, where $\lambda \in [0,1)$.

(ii) $\zeta(t,s) = \frac{s}{1+s} - t$ for all $t,s \in [0,+\infty)$.

(iii) $\zeta(t,s) = s - kt$ for all $t,s \in [0,+\infty)$, where $k > 1$.

(iv) $\zeta(t,s) = \frac{s}{1+s} - te^t$ for all $t,s \in [0,+\infty)$.

(v) $\zeta(t,s) = s - \varphi(s) - t$ for all $s,t \in [0,+\infty)$ where $\varphi : [0,+\infty) \to [0,+\infty)$ is a lower semi continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.

In 2012, Sedghi, Shobe and Aliouche [17] introduced S-metric spaced and studied their properties. The following are preliminaries on S-metric spaces.

Definition 1.3 ([17]). Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0,+\infty)$ that satisfies the following conditions: for each $x,y,z,a \in X$
(S1) $S(x,y,z) = 0$ if and only if $x = y = z$ and

(S2) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.
The pair (X,S) is called an S-metric space.

Throughout this paper, we denote the set of all real numbers by $\mathbb{R}$, the set of all natural numbers by $\mathbb{N}$.

Example 1.4 ([17]). Let (X,d) be a metric space. We define $S : X^3 \to [0,+\infty)$ by $S(x,y,z) = d(x,y) + d(x,z) + d(y,z)$ for all $x,y,z \in X$. Then S is an S-metric on X and S is called the S-metric induced by the metric d.

Example 1.5 ([9]). Let $X = \mathbb{R}$ and $S(x,y,z) = |y+z-2x|+|y-z|$ for all $x,y,z \in X$. Then (X,S) is an S-metric space.

Example 1.6 ([18]). Let $X = \mathbb{R}$. Then $S(x,y,z) = |x-z| + |y-z|$ for all $x,y,z \in \mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric is called the usual S-metric on $\mathbb{R}$.

Lemma 1.7 ([17]). *In an S-metric space, we have $S(x,x,y) = S(y,y,x)$.*

Lemma 1.8 ([9]). *Let (X,S) be an S-metric space. Then $S(x,x,z) \leq 2S(x,x,y) + S(y,y,z)$.*

Definition 1.9 ([17]). Let (X,S) be an S-metric space.
(i) A sequence $\{x_n\} \subseteq X$ is said to converge to a point $x \in X$ if $S(x_n,x_n,x) \to 0$ as $n \to +\infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n,x_n,x) < \epsilon$ and we denote it by $\lim\limits_{n\to+\infty} x_n = x$.

(ii) A sequence $\{x_n\} \subseteq X$ is called a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \epsilon$ for all $n,m \geq n_0$.

(iii) An S-metric space (X,S) is said to be complete if each Cauchy sequence in X is convergent.

LEMMA 1.10 ([17]). *Let (X,S) be an S-metric space. If $\{x_n\}$ is a sequence in X that converges to x, then x is unique.*

LEMMA 1.11 ([6]). *Let (X,S) be an S-metric space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $\{x_n\}$ converges to x in X. Then $\lim\limits_{n\to+\infty} S(x_n,x_n,y_n)=\lim\limits_{n\to+\infty} S(x,x,y_n)$.*

DEFINITION 1.12 ([17]). Let (X,S) be an S-metric space. A map $F : X \to X$ is said to be a contraction if there exists a constant $0 \le K < 1$ such that

$$S(Fx,Fx,Fy) \le KS(x,x,y), \quad \text{for all } x,y \in X. \tag{1}$$

THEOREM 1.13 ([17]). *Let (X,S) be a complete S-metric space and let a map $F : X \to X$ be a contraction. Then F has a unique fixed point u in X.*

LEMMA 1.14 ([8]). *Let (X,S) be an S-metric space and $\{x_n\}$ be a sequence in X such that $\lim_{n\to+\infty} S(x_n,x_n,x_{n+1}) = 0$.*

If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $m_k > n_k > k$ such that $S(x_{m_k},x_{m_k},x_{n_k}) \ge \epsilon$ with $S(x_{m_k-1},x_{m_k-1},x_{n_k}) < \epsilon$. Also, we have the following:

(i) $\lim\limits_{k\to+\infty} S(x_{m_k},x_{m_k},x_{n_k}) = \epsilon$ *(ii)* $\lim\limits_{k\to+\infty} S(x_{m_k-1},x_{m_k-1},x_{n_k}) = \epsilon$

(iii) $\lim\limits_{k\to+\infty} S(x_{m_k},x_{m_k},x_{n_k-1}) = \epsilon$ *(iv)* $\lim\limits_{k\to+\infty} S(x_{m_k-1},x_{m_k-1},x_{n_k-1}) = \epsilon$.

For more works on S-metric spaces we refer to [3, 6, 8, 9, 17, 18].

In 2019, Mlaiki, Yılmaz Özgür and Nihal Taş [14] introduced $\mathcal{Z}_s$-contraction by using simulation functions and proved the existence and uniqueness of fixed points of such mapping in complete S-metric spaces.

DEFINITION 1.15 ([14]). Let (X,S) be an S-metric space and $T : X \to X$. If there exists a $\zeta \in \mathcal{Z}$ such that

$$\zeta(S(Tx,Tx,Ty),S(x,x,y)) \ge 0 \tag{2}$$

for all $x,y \in X$, then T is called a $\mathcal{Z}_s$-contraction with respect to ζ.

THEOREM 1.16 ([14]). *Let (X,S) be a complete S-metric space and $T : X \to X$. If T is a $\mathcal{Z}_s$-contraction with respect to ζ, then T has a unique fixed point $a \in X$, and for every $x_0 \in X$ the sequence $\{x_n\}$ converges to a, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$.*

Motivated by the works of [10, 13–15], in Section 2 of this paper, we introduce almost Suzuki type $\mathcal{Z}_s$-contraction mappings and prove the existence and uniqueness of fixed points of such mappings in complete S-metric spaces. Our results generalize the fixed point theorems of Mlaiki, Yılmaz Özgür and Nihal Taş [14] and Sedghi, Shobe and Aliouche [17]. We draw some corollaries and give examples in support of our results.

2. Main results

In the following we define almost Suzuki type $\mathcal{Z}_s$-contraction with respect to a simulation function.

Definition 2.1. Let (X, S) be an S-metric space and $T : X \to X$. If there exist a $\zeta \in \mathcal{Z}$ and $L \geq 0$ such that

$$\frac{1}{3}S(x,x,Tx) < S(x,y,z) \text{ implies } \zeta(S(Tx,Ty,Tz), S(x,y,z)+LN(x,y,z)) \geq 0, \quad (3)$$

for all $x, y, z \in X$, where $N(x,y,z) = \min\{S(Tx,Tx,x), S(Tx,Tx,y), S(Tx,Tx,z), S(Tx,y,z)\}$, then T is called an almost Suzuki type $\mathcal{Z}_s$-contraction with respect to ζ.

The following theorem is the main result of this paper.

Theorem 2.2. *Let (X, S) be a complete S-metric space and $T : X \to X$ be an almost Suzuki type $\mathcal{Z}_s$-contraction with respect to $\zeta \in \mathcal{Z}$. Then for any $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_n = Tx_{n-1}$ for all $n = 1, 2, \dots$ is Cauchy in X, $\lim_{n\to+\infty} x_n = u$ (say) in X and u is a unique fixed point in X.*

Proof. Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined as $x_n = Tx_{n-1}$ for all $n = 1, 2, 3, \dots$. If $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ for some n_0, then x_{n_0} is a fixed point of T.

Therefore, we assume that $x_n \neq x_{n+1}$, i.e., $S(x_n, x_n, x_{n+1}) > 0$, for all $n \geq 0$.
STEP 1: We prove that $\lim_{n\to+\infty} S(x_n, x_n, x_{n+1}) = 0$.

We have $\frac{1}{3}S(x_n, x_n, Tx_n) < S(x_n, x_n, Tx_n) = S(x_n, x_n, x_{n+1})$, hence from the inequality (3), we have

$$0 \leq \zeta(S(Tx_n, Tx_n, Tx_{n+1}), S(x_n, x_n, x_{n+1}) + LN(x_n, x_n, x_{n+1})). \quad (4)$$

Here

$$\begin{aligned} N(x_n,x_n,x_{n+1}) &= \min\{S(Tx_n,Tx_n,x_n), S(Tx_n,Tx_n,x_{n+1}), S(Tx_n,x_n,x_{n+1})\} \\ &= \min\{S(x_{n+1},x_{n+1},x_n), S(x_{n+1},x_{n+1},x_{n+1}), S(x_{n+1},x_n,x_{n+1})\} = 0 \end{aligned}$$

From the inequality (4) and by using (ζ_2), we get

$$0 \leq \zeta(S(x_{n+1},x_{n+1},x_{n+2}), S(x_n,x_n,x_{n+1})) < S(x_n,x_n,x_{n+1}) - S(x_{n+1},x_{n+1},x_{n+2})$$

which implies that

$$S(x_{n+1},x_{n+1},x_{n+2}) < S(x_n,x_n,x_{n+1}), \text{ for all } n = 0,1,2,\dots . \quad (5)$$

Therefore, the sequence $\{S(x_n, x_n, x_{n+1})\}$ is decreasing and converges to some $r \geq 0$. Assume that $r > 0$.

Let $t_n = S(x_{n+1}, x_{n+1}, x_{n+2})$ and $s_n = S(x_n, x_n, x_{n+1})$. Since $\lim_{n\to+\infty} t_n = \lim_{n\to+\infty} s_n = r > 0$, by using the inequality (3) and the condition (ζ_3), we get that

$$0 \leq \limsup_{n\to+\infty} \zeta(S(x_{n+1},x_{n+1},x_{n+2}), S(x_n,x_n,x_{n+1})) < 0,$$

a contradiction. Therefore $r = 0$, i.e.

$$\lim_{n\to+\infty} S(x_n,x_n,x_{n+1}) = 0. \quad (6)$$

STEP 2: We prove that $\{x_n\}$ is a Cauchy sequence.

If $\{x_n\}$ is not Cauchy, by Lemma 1.14 there exist an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that $m_k > n_k \geq k$ such that $S(x_{m_k}, x_{m_k}, x_{n_k}) \geq \epsilon$ and $S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) < \epsilon$ satisfying (i)-(iv) of Lemma 1.14.

Now suppose, if possible, there exists a $k \geq k_1$ such that

$$\frac{1}{3}S(x_{m_k-1}, x_{m_k-1}, x_{m_k}) \geq S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}). \tag{7}$$

On letting $k \to +\infty$ in the inequality (7) and using Lemma 1.14 (iv), we get that $\epsilon \leq 0$, which is a contraction. Therefore $\frac{1}{3}S(x_{m_k-1}, x_{m_k-1}, x_{m_k}) < S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1})$, for all $k \geq k_1$. Now, we have

$$\begin{aligned}
&N(x_{m_k-1}, x_{m_k-1}, x_{n_k-1})\\
&= \min\{S(Tx_{m_k-1}, Tx_{m_k-1}, x_{m_k-1}), S(Tx_{m_k-1}, Tx_{m_k-1}, x_{n_k-1}),\\
&= \min\{S(x_{m_k}, x_{m_k}, x_{m_k-1}), S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(x_{m_k}, x_{m_k-1}, x_{n_k-1})\}.
\end{aligned}$$

By letting $k \to +\infty$ and by using the inequality (6), we get

$$\lim_{k\to+\infty} N(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) = 0. \tag{8}$$

Let $t'_k = S(x_{m_k}, x_{m_k}, x_{n_k})$ and $s'_k = S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) + LN(x_{m_k-1}, x_{m_k-1}, x_{n_k-1})$. By using Lemma 1.14 and the inequality (8), we obtain that $\lim\limits_{k\to+\infty} t'_k = \lim\limits_{k\to+\infty} s'_k = \epsilon > 0$.

Now, by the inequality (3) and by (ζ_3), we have

$$0 \leq \limsup_{k\to+\infty} \zeta(S(x_{m_k}, x_{m_k}, x_{n_k}), S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) + LN(x_{m_k-1}, x_{m_k-1}, x_{n_k-1})) < 0,$$

a contradiction.

Therefore, $\{x_n\}$ is a Cauchy sequence. Since (X, S) is a complete S-metric space, there exists a $u \in X$, such that $\lim\limits_{n\to+\infty} x_n = u$.

STEP 3: We now prove that u is a fixed point of T.

Suppose that $Tu \neq u$. Then $S(u, u, Tu) > 0$. We now prove that either

(a) $$\frac{1}{3}S(x_n, x_n, x_{n+1}) < S(x_n, x_n, u), \quad \text{or}$$

(b) $$\frac{1}{3}S(x_{n+1}, x_{n+1}, x_{n+2}) < S(x_{n+1}, x_{n+1}, u) \tag{9}$$

hold for each $n = 0, 1, 2, \ldots$. On the contrary, suppose that $\frac{1}{3}S(x_n, x_n, x_{n+1}) \geq S(x_n, x_n, u)$ and $\frac{1}{3}S(x_{n+1}, x_{n+1}, x_{n+2}) \geq S(x_{n+1}, x_{n+1}, u)$ hold for some n. Therefore, $3S(x_n, x_n, u) \leq S(x_n, x_n, x_{n+1}) = S(x_{n+1}, x_{n+1}, x_n) \leq 2S(x_{n+1}, x_{n+1}, u) + S(x_n, x_n, u)$, which implies that $S(x_n, x_n, u) \leq S(x_{n+1}, x_{n+1}, u)$. From the inequality (5), we have

$$\begin{aligned}
S(x_{n+1}, x_{n+1}, x_{n+2}) &< S(x_n, x_n, x_{n+1})\\
&\leq 2S(x_n, x_n, u) + S(x_{n+1}, x_{n+1}, u) \leq S(x_{n+1}, x_{n+1}, x_{n+2}),
\end{aligned}$$

a contradiction. Therefore (9) holds.

Suppose that (a) holds, i.e. $\frac{1}{3}S(x_n, x_n, Tx_n) < S(x_n, x_n, u)$ and from (3) we have

$$0 \leq \zeta(S(Tx_n, Tx_n, Tu), S(x_n, x_n, u) + LN(x_n, x_n, u)). \tag{10}$$

Here

$$\begin{aligned} N(x_n, x_n, u) &= \min\{S(Tx_n, Tx_n, x_n), S(Tx_n, Tx_n, u), S(Tx_n, x_n, u)\} \\ &= \min\{S(x_{n+1}, x_{n+1}, x_n), S(x_{n+1}, x_{n+1}, u), S(x_{n+1}, x_n, u)\}. \end{aligned}$$

On letting $n \to +\infty$, we have $\lim_{n\to+\infty} N(x_n, x_n, u) = 0$. From (10), we have

$$\begin{aligned} 0 &\leq \limsup_{n\to+\infty} \zeta(S(x_{n+1}, x_{n+1}, Tu), S(x_n, x_n, u) + LN(x_n, x_n, u)) \\ &\leq \limsup_{n\to+\infty} (S(x_n, x_n, u) + LN(x_n, x_n, u) - S(x_{n+1}, x_{n+1}, Tu)) \\ &= \limsup_{n\to+\infty} (S(x_n, x_n, u) + LN(x_n, x_n, u)) - \liminf_{n\to+\infty} S(x_{n+1}, x_{n+1}, Tu) = -S(u, u, Tu), \end{aligned}$$

Hence $S(u, u, Tu) = 0$, i.e., $u = Tu$.

Now suppose (b) holds, i.e. $\frac{1}{3}S(x_{n+1}, x_{n+1}, Tx_{n+1}) < S(x_{n+1}, x_{n+1}, u)$ and from the inequality (3) we have

$$0 \leq \zeta(S(Tx_{n+1}, Tx_{n+1}, Tu), S(x_{n+1}, x_{n+1}, u) + LN(x_{n+1}, x_{n+1}, u)), \tag{11}$$

where

$$\begin{aligned} N(x_{n+1}, x_{n+1}, u) &= \min\{S(Tx_{n+1}, Tx_{n+1}, x_{n+1}), S(Tx_{n+1}, Tx_{n+1}, u), S(Tx_{n+1}, x_{n+1}, u)\} \\ &= \min\{S(x_{n+2}, x_{n+2}, x_{n+1}), S(x_{n+2}, x_{n+2}, u), S(x_{n+2}, x_{n+1}, u)\}. \end{aligned}$$

On letting $n \to +\infty$, we have $\lim_{n\to+\infty} N(x_{n+1}, x_{n+1}, u) = 0$. From the inequality (11), we have

$$\begin{aligned} 0 &\leq \limsup_{n\to+\infty} \zeta(S(x_{n+2}, x_{n+2}, Tu), S(x_{n+1}, x_{n+1}, u) + LN(x_{n+1}, x_{n+1}, u) \\ &\leq \limsup_{n\to+\infty} (S(x_{n+1}, x_{n+1}, u) + LN(x_{n+1}, x_{n+1}, u) - S(x_{n+2}, x_{n+2}, Tu)) = -S(u, u, Tu). \end{aligned}$$

Hence $S(u, u, Tu) = 0$, i.e., $u = Tu$. Thus, u is a fixed point of T.

STEP 4: We now prove the uniqueness. Suppose that $x, y \in X$ are two fixed points of T such that $x \neq y$. Clearly, $\frac{1}{3}S(x, x, Tx) < S(x, x, y)$. Then by using the inequality (3), we get

$$0 \leq \zeta(S(Tx, Tx, Ty), S(x, x, y) + LN(x, x, y)), \tag{12}$$

where $N(x, x, y) = \min\{S(Tx, Tx, x), S(Tx, Tx, y), S(Tx, x, y)\} = 0$. Now, from the inequality (12), we get

$$0 \leq \zeta(S(x, x, y), S(x, x, y)) < S(x, x, y) - S(x, x, y) = 0 \qquad \text{(by } (\zeta_2)\text{)},$$

a contradiction. Therefore $x = y$. $\square$

THEOREM 2.3. *Let (X, S) be a complete S-metric space. Suppose that there exist $\zeta \in \mathcal{Z}$ and $L \geq 0$ such that*

$$\frac{1}{3}S(x, x, Tx) < S(x, x, y) \quad \textit{implies} \quad \zeta(S(Tx, Tx, Ty), S(x, x, y) + LN(x, x, y)) \geq 0, \tag{13}$$

for all $x, y \in X$, where $N(x, x, y) = \min\{S(Tx, Tx, x), S(Tx, Tx, y), S(Tx, x, y)\}$.

Then T has a unique fixed point in X.

Proof. Similar to the proof of Theorem 2.2 $\square$

Corollary 2.4. *Let (X,S) be a complete S-metric space. Suppose that there exist $\zeta \in \mathcal{Z}$ and $L \geq 0$ and such that*

$$\zeta(S(Tx,Tx,Ty),S(x,x,y)+LN(x,x,y)) \geq 0, \tag{14}$$

for all $x,y \in X$, where $N(x,x,y)$ is defined as in the inequality (13). *Then T has a unique fixed point in X.*

Remark 2.5. If $L=0$ in the inequality (13), then Theorem 1.16 follows as a corollary of Corollary 2.4.

Corollary 2.6. *Let (X,S) be a complete S-metric space and $T: X\times X \to X$ be a mapping satisfying the following condition: there exists $\lambda \in [0,1)$ such that*

$$\frac{1}{3}S(x,x,Tx)<S(x,x,y) \text{ implies } S(Tx,Tx,Ty)\leq\lambda S(x,x,y), \tag{15}$$

for all $x,y \in X$. Then T has a unique fixed point in X.

Proof. If we choose simulation function ζ as $\zeta(t,s)=\lambda s-t$, for all $s,t\geq 0$, where $\lambda \in [0,1)$ and if $L=0$ in the inequality (13) then the inequality (15) is a special case of the inequality (13) so that, the conclusion of this corollary follows from Theorem 2.3. $\square$

Remark 2.7. Theorem 1.13 follows as a corollary of Corollary 2.6.

Corollary 2.8. *Let (X,S) be a complete S-metric space and $T: X\times X \to X$ be a mapping satisfying*

$$\frac{1}{3}S(x,x,Tx) < S(x,y,z) \text{ implies } S(Tx,Ty,Tz) \leq S(x,y,z)-\varphi(S(x,y,z)), \tag{16}$$

for all $x,y,z \in X$, where $\varphi:[0,+\infty)\to[0,+\infty)$ is a lower semi continuous function with $\varphi(t)=0$ if and only if $t=0$. Then T has a unique fixed point in X.

Proof. We choose $\zeta(t,s)$ as in the Example 1.2 (v) and if $L=0$ in the inequality (3), it follows that the conclusion of this corollary holds by applying Theorem 2.2. $\square$

The following example is in support of Theorem 2.2.

Example 2.9. Let $X=[\frac{1}{4},1]$. We define $S: X^3 \to [0,+\infty)$ by

$$S(x,y,z)=\begin{cases} 0, & \text{if } \ x=y=z \\ \max\{x,y,z\}, & \text{otherwise.}\end{cases}$$

We define $T: X\to X$ by

$$Tx=\begin{cases} \frac{1}{2}, & \text{if } \ x\in[\frac{1}{4},\frac{1}{2}) \\ 1, & \text{if } \ x\in[\frac{1}{2},1].\end{cases}$$

We define $\zeta:[0,+\infty)\times[0,+\infty)\to\mathbb{R}$ by $\zeta(t,s)=\frac{1}{2}s-t,\ s,t\geq 0$. Then ζ is a simulation function. Let $x,y,z\in X$. We now verify that $\zeta(S(Tx,Ty,Tz),S(x,y,z)+LN(x,y,z))\geq 0$ whenever $\frac{1}{3}S(x,x,Tx)<S(x,y,z)$.

Case (i) Let $x, y, z \in [\frac{1}{4}, \frac{1}{2})$. We assume without loss of generality that $x < y < z$. We have $\frac{1}{3}S(x, x, Tx) = \frac{1}{3}S(x, x, \frac{1}{2}) = \frac{1}{6} < z = S(x, y, z)$. In this case $S(Tx, Ty, Tz) = 0$ so that the inequality (3) holds trivially for any $L \geq 0$.

Case (ii) Let $x, y, z \in [\frac{1}{2}, 1]$. We assume that $x < y < z$. We have $\frac{1}{3}S(x, x, Tx) = \frac{1}{3}S(x, x, 1) = \frac{1}{3} < z = S(x, y, z)$. In this case $S(Tx, Ty, Tz) = 0$ so that the inequality (3) holds trivially for any $L \geq 0$.

Case (iii) Let $x \in [\frac{1}{4}, \frac{1}{2})$ and $y, z \in [\frac{1}{2}, 1]$. We assume that $y < z$. We have $\frac{1}{3}S(x, x, Tx) = \frac{1}{3}S(x, x, \frac{1}{2}) = \frac{1}{6} < z = S(x, y, z)$.

$$S(Tx, Tx, x) = S(\frac{1}{2}, \frac{1}{2}, x) = \frac{1}{2}; \qquad S(Tx, Tx, y) = S(\frac{1}{2}, \frac{1}{2}, y) = y$$

$$S(Tx, Tx, z) = S(\frac{1}{2}, \frac{1}{2}, z) = z; \qquad S(Tx, y, z) = S(\frac{1}{2}, y, z) = z.$$

$$N(x, y, z) = \min\{\frac{1}{2}, y, z\} = \frac{1}{2} \quad \text{and} \quad S(Tx, Ty, Tz) = S(\frac{1}{2}, 1, 1) = 1$$

We consider

$$\begin{aligned}\zeta(S(Tx, Ty, Tz), S(x, y, z) + LN(x, y, z) &= \frac{1}{2}(S(x, y, z) + LN(x, y, z)) - S(Tx, Ty, Tz) \\ &\geq \frac{1}{2}(LN(x, y, z)) - S(Tx, Ty, Tz) = \frac{1}{2}(L(\frac{1}{2})) - 1 \geq 0, \text{ for any } L \geq 4.\end{aligned}$$

In this case, the inequality (3) holds for any $L \geq 4$.

Case (iv) Let $x, y \in [\frac{1}{4}, \frac{1}{2})$ and $z \in [\frac{1}{2}, 1]$. We assume that $x < y$. We have $\frac{1}{3}S(x, x, Tx) = \frac{1}{3}S(x, x, \frac{1}{2}) = \frac{1}{6} < z = S(x, y, z)$.

$$S(Tx, Tx, x) = S(\frac{1}{2}, \frac{1}{2}, x) = \frac{1}{2} \qquad S(Tx, Tx, y) = S(\frac{1}{2}, \frac{1}{2}, y) = \frac{1}{2};$$

$$S(Tx, Tx, z) = S(\frac{1}{2}, \frac{1}{2}, z) = z \qquad S(Tx, y, z) = S(\frac{1}{2}, y, z) = z.$$

$$N(x, y, z) = \min\{\frac{1}{2}, z\} = \frac{1}{2} \quad \text{and} \quad S(Tx, Ty, Tz) = S(\frac{1}{2}, 1, 1) = 1.$$

In this case, we have $\zeta(S(Tx, Ty, Tz), S(x, y, z) + LN(x, y, z)) \geq 0$ for any $L \geq 4$ (similarly as in Case (iii)). The inequality (3) holds for any $L \geq 4$.

Case (v) Let $x, z \in [\frac{1}{4}, \frac{1}{2})$ and $y \in [\frac{1}{2}, 1]$. We assume that $x < z$. We have $\frac{1}{3}S(x, x, Tx) = \frac{1}{3}S(x, x, \frac{1}{2}) = \frac{1}{6} < y = S(x, y, z)$.

$$S(Tx, Tx, x) = S(\frac{1}{2}, \frac{1}{2}, x) = \frac{1}{2}; \qquad S(Tx, Tx, y) = S(\frac{1}{2}, \frac{1}{2}, y) = y;$$

$$S(Tx, Tx, z) = S(\frac{1}{2}, \frac{1}{2}, z) = \frac{1}{2}; \qquad S(Tx, y, z) = S(\frac{1}{2}, y, z) = y.$$

$$N(x, y, z) = \min\{\frac{1}{2}, y\} = \frac{1}{2} \quad \text{and} \quad S(Tx, Ty, Tz) = 1.$$

Similarly as in Case (iii), the inequality (3) holds for any $L \geq 4$.

Case (vi) Let $z \in [\frac{1}{4}, \frac{1}{2})$ and $x, y \in [\frac{1}{2}, 1]$. We assume that $x < y$. We have

$\frac{1}{3}S(x,x,Tx) = \frac{1}{3}S(x,x,1) = \frac{1}{3} < y = S(x,y,z)$.

$$S(Tx,Tx,x) = S(1,1,x) = 1; \qquad S(Tx,Tx,y) = S(1,1,y) = 1;$$
$$S(Tx,Tx,z) = S(1,1,z) = 1; \qquad S(Tx,y,z) = S(1,y,z) \text{ so that}$$
$$N(x,y,z) = 1 \quad \text{and} \qquad S(Tx,Ty,Tz) = 1.$$

We consider

$$\zeta(S(Tx,Ty,Tz), S(x,y,z) + LN(x,y,z)) = \frac{1}{2}(S(x,y,z) + LN(x,y,z)) - S(Tx,Ty,Tz)$$
$$\geq \frac{1}{2}(LN(x,y,z)) - S(Tx,Ty,Tz) = \frac{1}{2}(L(1)) - 1 \geq 0, \text{ for any } L \geq 2.$$

In this case, the inequality (3) holds for any $L \geq 2$.

Case (vii) Let $y \in [\frac{1}{4}, \frac{1}{2})$ and $x, z \in [\frac{1}{2}, 1]$. We assume that $x < z$. We have $\frac{1}{3}S(x,x,Tx) = \frac{1}{3}S(x,x,1) = \frac{1}{3} < z = S(x,y,z)$.

$$S(Tx,Tx,x) = S(1,1,x) = 1; \qquad S(Tx,Tx,y) = S(1,1,y) = 1;$$
$$S(Tx,Tx,z) = S(1,1,z) = 1; \qquad S(Tx,y,z) = S(1,y,z) = 1 \text{ so that}$$
$$N(x,y,z) = 1.$$

In this case, we have $\zeta(S(Tx,Ty,Tz), S(x,y,z) + LN(x,y,z)) \geq 0$, for any $L \geq 2$ (similarly as in Case (vi)).

Case (viii) Let $z, y \in [\frac{1}{4}, \frac{1}{2})$ and $x \in [\frac{1}{2}, 1]$. We assume that $y < z$. We have $\frac{1}{3}S(x,x,Tx) = \frac{1}{3}S(x,x,1) = \frac{1}{3} < S(x,y,z)$. Here $N(x,y,z) = 1$. In this case, we have $\zeta(S(Tx,Ty,Tz), S(x,y,z) + LN(x,y,z)) \geq \frac{1}{2}(L(1)) - 1 \geq 0$, for any $L \geq 2$. Therefore T satisfies all the hypotheses of Theorem 2.2, with $L = 4$, and 1 is the unique fixed point of T.

The following example is in support of Theorem 2.3.

Example 2.10. Let $X = [\frac{1}{2}, 2]$. We define $S : X^3 \to [0, \infty)$ by

$$S(x,y,z) = \begin{cases} 0, & \text{if } x = y = z \\ \max\{x,y,z\}, & \text{otherwise.} \end{cases}$$

Then (X, S) is a complete S-metric space. We define $T : X \to X$ by

$$Tx = \begin{cases} \frac{2x+1}{3}, & \text{if } x \in [\frac{1}{2}, 1) \\ 2, & \text{if } x \in [1, 2]. \end{cases}$$

We define $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\zeta(t,s) = s - 3t$, $s, t \geq 0$. Then ζ is a simulation function. Let $x, y \in X$. We now verify that $\zeta(S(Tx,Tx,Ty), S(x,x,y) + LN(x,x,y)) \geq 0$ whenever $\frac{1}{3}S(x,x,Tx) < S(x,x,y)$.

Case (i) Let $x, y \in [\frac{1}{2}, 1)$ with $x < y$. Here $S(x,x,Tx) = \frac{2x+1}{3}$ and $S(x,x,y) = y$. Clearly, $\frac{1}{3}S(x,x,Tx) < S(x,x,y)$. In this case, $S(Tx,Tx,Ty) = \frac{2y+1}{3}$ and $N(x,y,z) = \frac{2x+1}{3}$. Then

$$\zeta(S(Tx,Tx,Ty), S(x,x,y) + LN(x,x,y)) = S(x,x,y) + LN(x,x,y) - 3S(Tx,Tx,Ty)$$
$$= y + L(\frac{2x+1}{3}) - 3(\frac{2y+1}{3}) > 0 \quad \text{for any } L \geq 4.$$

Therefore, in this case, the inequality (13) holds for any $L \geq 4$.

Case (ii) Let $x, y \in [1, 2]$. We have $\frac{1}{3}S(x, x, Tx) = \frac{1}{3}S(x, x, 2) = \frac{2}{3} < S(x, x, y)$. In this case $S(Tx, Tx, Ty) = 0$ so that the inequality (13) holds trivially for any $L \geq 0$.

Case (iii) Let $x \in [\frac{1}{2}, 1)$ and $y \in [1, 2]$. We have $\frac{1}{3}S(x, x, Tx) = \frac{1}{3}\max\{\frac{2x+1}{3}, x\} < S(x, x, y)$. Here $S(Tx, Tx, Ty)$=2; $S(Tx, Tx, x)$=$\frac{2x+1}{3}$; $S(Tx, Tx, y)$=y; $S(Tx, x, y)$=y and $N(x, x, y)$= $\min\{\frac{2x+1}{3}, y\}$=$\frac{2x+1}{3}$. We consider

$$\zeta(S(Tx, Tx, Ty), S(x, x, y) + LN(x, x, y)) = y + L(\frac{2x+1}{3}) - 3(2) > 0$$

for any $L \geq 9$. Therefore, in this case, the inequality (13) holds for any $L \geq 9$.

Case (iv) Let $x \in [1, 2]$ and $y \in [\frac{1}{2}, 1)$. We have $\frac{1}{3}S(x, x, Tx) = \frac{2}{3} < S(x, x, y)$ and $N(x, x, y) = 2$; We consider $\zeta(S(Tx, Tx, Ty), S(x, x, y) + LN(x, x, y)) = x + L(2) - 3(2) > 0$ for any $L \geq 3$. In this case, the inequality (13) holds for any $L \geq 3$.

Therefore T satisfies all the hypotheses of Theorem 2.3 with respect to ζ with $L = 9$ and 2 is the unique fixed point of T.

Here we observe that T does not satisfy the inequality (1). For example, choose $x = \frac{3}{2}$, $y = \frac{3}{4}$, we have $S(x, x, y) = \frac{3}{2}$. Now, $S(Tx, Tx, Ty) = 2 \nleq KS(x, x, y)$ for any $K < 1$. Hence T does not satisfy the inequality (1). So we conclude from Remark 2.7 that Theorem 2.3 generalizes Theorem 1.13.

Further, we observe that T does not satisfy the inequality (2) for any ζ. If we choose $x = \frac{1}{2}$, $y = \frac{1}{4}$ in the inequality (2), then we have $\zeta(S(Tx, Tx, Ty), S(x, x, y)) = \zeta(\frac{2}{3}, \frac{1}{2}) < \frac{1}{2} - \frac{2}{3} < -\frac{1}{6}$, a contradiction.

Thus the inequality (2) fails to hold for any ζ. Hence by Remark 2.5, it follows that Theorem 2.3 is a generalization of Theorem 1.16.

References

[1] M. Aggarwal, R. Chugh, R. Kamal, *Suzuki-type fixed point results in G-metric spaces and applications*, Int. J. Comput. Appl., **47(12)** (2012), 4 pages.

[2] M. A. Alghamdi, S. Gulyaz-Ozyurt, E. Karapınar, *A note on extended Z-contraction*, Mathematics, **8(195)** (2020), 14 pages.

[3] G. V. R. Babu, L. Bekere Kumssa, *Fixed points of (α, ψ, φ)-generalized weakly contractive maps and Property (P) in S-metric spaces*, Filomat, **31(14)** (2017), 4469–4481.

[4] G. V. R. Babu, Dula Tolera Mosissa, Fixed points in b-metric spaces via simulation function, Novi Sad J. Math., **47(2)** (2017), 133–147.

[5] G. V. R. Babu, D. Ratna Babu, *Fixed points of Suzuki $\mathcal{Z}$-contraction type maps in b-metric spaces*, Adv. Theory Nonlinear Anal. Appl., **4(1)** (2020), 14–28.

[6] G. V. R. Babu, P. Durga Sailaja, G. Srichandana, *Common fixed points via C_k-class functions in S-metric spaces*, J. Fixed Point Theory, **2020(1)** (2020), 22 pages.

[7] S. Chandok, Ankush Chanda, Lakshmi Kanta Dey, M. Pavlović, S. Radenović, *Simulation functions and Geraghty type results*, Bol. Soc. Paran. Mat., (3s.) **39(1)** (2021), 35–50.

[8] T. Došenović, S. Radenović, A. Rezvani, S. Sedghi, *Coincidence point theorems in S-metric spaces using inegral type of contraction*, U. P. B. Sci. Bull, Series A, **79(4)** (2017), 145–158.

[9] N. V. Dung, N. T. Hieu, S. Radojević, *Fixed point theorems for g-monotone maps on partially ordered S-metric spaces*, Filomat, **28(9)** (2014), 1885–1898.

[10] U. C. Gairola, Deepak Khantwal, *Suzuki type fixed point theorems in S-metric space*, IJMAA, **5(3-C)** (2017), 277–289.

[11] H. Isik, N. B. Gungor, C. Park, S. Y. Jang, *Fixed point theorems for almost Z-contractions with an application*, Mathematics, **6(37)** (2018), 8 pages.

[12] F. Khojasteh, S. Shukla, S. Radenović, *A new approach to the study of fixed point theory for simulation functions*, Filomat, **29(6)** (2015), 1189–1194.

[13] P. Kumam, D. Gopal, L. Budhia, *A new fixed point theorem under Suzuki type $\mathcal{Z}$-contraction mappings*, J. Math. Anal., **8(1)** (2017), 113–119.

[14] N. Mlaiki , N. Yılmaz Özgür, Nihal Taş, *New fixed-point theorems on an S-metric space via simulation functions*, Mathematics, **7(583)** (2019), 13 pages.

[15] A. Padcharoen, P. Kumam, P. Saipara, P. Chaipunya, *Generalized Suzuki type $\mathcal{Z}$-contraction in complete metric spaces*, Kragujevac J. Math., **42(3)** (2018), 419–430.

[16] S. Radenović, S. Chandok, *Simulation type functions and coincidence points*, Filomat, **32(1)** (2018), 141–147.

[17] S. Sedghi, N. Shobe, A. Aliouche, *A generalization of fixed point theorems in S-metric spaces*, Mat. Vesnik, **64(3)** (2012), 258–266.

[18] S. Sedghi, N. V. Dung, *Fixed point theorems on S-metric spaces*, Mat. Vesnik, **66(1)** (2014), 113–124.

[19] T. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness*, Proc. Amer. Math. Soc., **136(5)** (2008), 1861–1869.

[20] Xiao-Lan Liu, A. H. Ansari, S. Chandok, S. Radenović, *On some results in metric spaces using auxiliary simulation functions via new functions*, J. Comput. Anal. Appl., **24(6)** (2018), 1103–1114.

(received 10.03.2021; in revised form 29.04.2021; available online 04.02.2022)

Department of Mathematics, Andhra University, Visakhapatnam-530 003, India

E-mail: gvr_babu@hotmail.com

Department of Mathematics, Lendi Institute of Engineering and Technology, Vizianagaram-535 005, India

E-mail: sailajadurga@yahoo.com

Department of Mathematics, Andhra University, Visakhapatnam-530 003, India

Department of Mathematics, Satya Institute of Technology and Management, Vizianagaram-535 002, India

E-mail: sri.chandan3@gmail.com