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NEW FIXED POINT RESULTS FOR ASYMPTOTIC CONTRACTIONS AND ITS APPLICATION TO CANTILEVER BEAM PROBLEMS

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Abstract. In this article, we deal with some interesting variants of asymptotic contractions, namely Reich type and Chatterjea type weak asymptotic contractions defined on the usual metric spaces. We derive a couple of fixed point results concerning such contractions. Moreover, we look over the existence of solutions to a fourth-order two-point boundary value problem which is a particular type of cantilever beam problems. Furthermore, we construct numerical examples to justify our obtained results.

1. Introduction and preliminaries

The fixed point theory based on asymptotic contractions revolves about the assumptions on the iterations of the corresponding mapping. In fact, the notion of asymptotic contractions was originally proposed in connection with one of the initial extensions of Banach contraction principle due to Caccioppoli [5]. It states that for a self-map f defined on a complete metric space X, the Picard iteration converges to the unique fixed point of f, given that for each $n \geq 1$, there is a non-negative constant c_n such that $d(f^n(x), f^n(y)) \leq c_n d(x, y)$, holds for all $x, y \in X$, satisfying $\sum_{n=1}^{\infty} c_n < \infty$.

Now we recollect the definition of an asymptotic contraction due to Kirk [10], which constitutes a wider collection of mappings than the class of aforementioned mappings.

Definition 1.1. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be any contractive gauge function such that φ is continuous and $\varphi(s) < s$ for $s > 0$. Let Φ be the collection of all such contractive gauge functions φ. A self-map f defined on a metric space (X, d) is said to be an asymptotic contraction if for all $n \in \mathbb{N}$, $d(f^n(x), f^n(y)) \leq \varphi_n(d(x, y))$ holds for all

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$x, y \in X$, where $\varphi_n : [0,1) \to [0,1)$ are functions such that $\varphi_n \to \varphi \in \Phi$ uniformly on the range of d.

Afterwards, Xu [16] introduced the idea of a weak asymptotic contraction and in the following, we note down the definition of such contractions.

Definition 1.2. A continuous self-map f defined on a metric space (X, d) is said to be a weakly asymptotic contraction if for an arbitrary $\epsilon > 0$, there exists an integer $n_\epsilon > 1$ such that $d(f^{n_\epsilon}(x), f^{n_\epsilon}(y)) \leq \varphi(d(x, y)) + \epsilon$ holds for all $x, y \in X$, where $\varphi \in \Phi$.

After this, Goyal [6] obtained a generalization of weak asymptotic contraction replacing the continuity of the contractive gauge function φ by upper semi-continuity. The author also derived a fixed point result concerning such generalization. Here it is important to emphasize that in the fixed point results of Xu [16] and Goyal [6] related to weak asymptotic contractions, it is assumed that the underlying mapping T is continuous.

Following this direction of research, different notions related to such kind of contractions were investigated in numerous ways in several underlying structures and consequently, a number of interesting results can be found in [1–3, 7, 11, 13, 14, 16] and references therein. Recently, Bera et al. [3] coined the concepts of asymptotic contraction in pair for two mappings and asymptotic contraction in pair for a finite number of mappings. The authors also obtained some interesting common fixed point theorems involving these asymptotic contractions in pair notions, and applied these findings to confirm the unique common solution to a particular type of pair of matrix equations. Continuing this line of study, in this sequel, following the notion of weak asymptotic contractions due to Goyal [6], we bring out the concepts of Reich and Chatterjea type weak asymptotic contractions on usual metric spaces. Also, we secure a few fixed point results involving such contractions without assuming the continuity of the underlying mappings. Additionally, our findings are endorsed by suitable constructive numerical examples.

On the other hand, the fixed point theory has enrolled plenty of researchers in finding new results not only from the theoretical point of view but also in different applicable areas, for example see [3, 4, 8, 9, 12, 15]. As an application of our obtained results, in this article, we enquire for sufficient conditions for the existence of solutions to a certain type of fourth-order two-point boundary value problem which is otherwise called a cantilever beam problem [12].

Before proceeding to our main results, we recall the following definitions.

Definition 1.3. Let T be a self-mapping on a metric space (X, d). Then the orbit of T at $x \in X$ is defined as the set $O_x(T) = \{x, Tx, T^2x, T^3x, \dots\}$. The mapping T is said to have a bounded orbit at $x \in X$ if the set $O_x(T)$ is bounded in the metric space (X, d).

Definition 1.4. Let T be a self-mapping on a metric space (X, d). Then T is said to be orbitally continuous if for any $x \in X$ and for any sequence (y_n) in $O_x(T)$, $y_n \to u$ implies $Ty_n \to Tu$ as $n \to \infty$.

2. Main results

At first, we consider a collection of functions Φ_1 which contains all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ which are upper semi-continuous and satisfy $\varphi(s) < s$ for $s > 0$ and $\varphi(0) = 0$.

For any function $\varphi \in \Phi_1$, we define the function $\tilde{\varphi}$ by

$$\tilde{\varphi}(t) = \max\{\varphi(s) : s \in [0, t] \cap \overline{R(d)}\}, \quad \text{where } R(d) \text{ is the range of } d.$$

The subsequent properties of $\tilde{\varphi}$ are due to Goyal [6]:

(a) $\tilde{\varphi}$ is upper semi-continuous;

(b) $\tilde{\varphi}(t) < t$ for all $t > 0$;

(c) $\tilde{\varphi}$ is increasing.

Now we propose the concepts of Reich type and Chatterjea type weakly asymptotic contractions.

Definition 2.1. Let (X, d) be a metric space and let T be a self-map defined on X. Suppose that for any $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that

$$d(T^{n_\epsilon}x, T^{n_\epsilon}y) \leq \varphi\left(ad(x, y) + bd(x, Tx) + cd(y, Ty)\right) + \epsilon$$

holds for all $x, y \in X$, where $\varphi \in \Phi_1$ and $a, b, c \geq 0$ and $a + b + c \leq 1$. Then T is said to be a Reich type weakly asymptotic contraction.

Definition 2.2. Let (X, d) be a metric space and let T be a self-map defined on X. Suppose that for any $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that

$$d(T^{n_\epsilon}x, T^{n_\epsilon}y) \leq \varphi(ad(x, y) + bd(x, Ty) + cd(y, Tx)) + \epsilon$$

holds for all $x, y \in X$, where $\varphi \in \Phi_1$ and $a, b, c \geq 0$ and $a + b + c \leq 1$. Then T is said to be a Chatterjea type weakly asymptotic contraction.

The subsequent fixed point result is related to the newly introduced notion of Reich type weakly asymptotic contraction.

Theorem 2.3. *Let (X, d) be a complete metric space and T be a Reich type weakly asymptotic contraction. Further, assume that T has a bounded orbit at $x \in X$ and T is orbitally continuous. Then T has a unique fixed point and the sequence $(T^n x)$ converges to that fixed point.*

Proof. Let us choose an arbitrary but fixed element $x \in X$ and construct a sequence (x_n) defined by $x_n = T^n x$ for all $n \in \mathbb{N}$. Since T has a bounded orbit, we have $D = \limsup_{n\to\infty} d(x_n, x_{n+1}) < \infty$.

Let $\epsilon > 0$ be arbitrary. Then there exists $n_\epsilon \in \mathbb{N}$ such that for all $x, y \in X$ we have

$$\begin{aligned} d(T^{n_\epsilon}x_n, T^{n_\epsilon}x_{n+1}) \leq& \varphi(ad(x_n, x_{n+1}) + bd(x_n, x_{n+1}) + cd(x_{n+1}, x_{n+2})) + \epsilon \\ \leq& \tilde{\varphi}(ad(x_n, x_{n+1}) + bd(x_n, x_{n+1}) + cd(x_{n+1}, x_{n+2})) + \epsilon. \end{aligned}$$

Further, we know that

$$\limsup_{n\to\infty} [ad(x_n, x_{n+1}) + bd(x_n, x_{n+1}) + cd(x_{n+1}, x_{n+2})] = aD + bD + cD. \tag{1}$$

Now, as $\tilde{\varphi}$ is upper semi-continuous,

$$\limsup_{n\to\infty} \tilde{\varphi}(ad(x_n, x_{n+1}) + bd(x_n, x_{n+1}) + cd(x_{n+1}, x_{n+2})) \leq \tilde{\varphi}((a+b+c)D). \quad (2)$$

Therefore from (1) and (2), we have

$$\limsup_{n\to\infty} d(T^{n_\epsilon}x_n, T^{n_\epsilon}x_{n+1}) \leq \tilde{\varphi}((a+b+c)D) + \epsilon \quad \Rightarrow D \leq \tilde{\varphi}((a+b+c)D) + \epsilon.$$

Since $\epsilon > 0$ and $(a+b+c) \leq 1$, we have $D = 0$, i.e., $\limsup_{n\to\infty} d(x_n, x_{n+1}) = 0$. Further, we consider $d_{n,m} = d(T^n x, T^m x)$ for all $n, m \geq 1$ and

$$d_\infty = \limsup_{n,m\to\infty} d_{n,m} = \limsup_{k\to\infty} \{d_{n,m} : n, m \geq k\}.$$

Let $\epsilon > 0$ be arbitrary and so there exists an $n_\epsilon \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$

$$\begin{aligned}
d_{n,m} =& d(T^n x, T^m x) = d(T^{n_\epsilon}(T^{n-n_\epsilon}x), T^{n_\epsilon}(T^{m-n_\epsilon}x)) \\
\leq& \varphi(ad(T^{n-n_\epsilon}x, T^{m-n_\epsilon}x) + bd(T^{n-n_\epsilon}x, T(T^{n-n_\epsilon}x)) + cd(T^{m-n_\epsilon}x, T(T^{m-n_\epsilon}x))) + \epsilon \\
=& \varphi(ad(x_{n-n_\epsilon}, x_{m-n_\epsilon}) + bd(x_{n-n_\epsilon}, x_{n-n_\epsilon+1}) + cd(x_{m-n_\epsilon}, x_{m-n_\epsilon+1})) + \epsilon \\
\leq& \tilde{\varphi}(ad(x_{n-n_\epsilon}, x_{m-n_\epsilon}) + bd(x_{n-n_\epsilon}, x_{n-n_\epsilon+1}) + cd(x_{m-n_\epsilon}, x_{m-n_\epsilon+1})) + \epsilon \\
d_{n,m} \leq& \tilde{\varphi}(ad(x_{n-n_\epsilon}, x_{m-n_\epsilon}) + bd(x_{n-n_\epsilon}, x_{n-n_\epsilon+1}) + cd(x_{m-n_\epsilon}, x_{m-n_\epsilon+1})) + \epsilon. \quad (3)
\end{aligned}$$

Again, we have

$$\limsup_{n,m\to\infty} (ad_{n-n_\epsilon,m-n_\epsilon} + bd_{n-n_\epsilon,n-n_\epsilon+1} + cd_{m-n_\epsilon,m-n_\epsilon+1}) = ad_\infty$$

$$\Rightarrow \limsup_{n,m\to\infty} ad_{n-n_\epsilon,m-n_\epsilon} = ad_\infty \;\Rightarrow \limsup_{k\to\infty} \{ad_{n-n_\epsilon,m-n_\epsilon} : n-n_\epsilon, m-n_\epsilon \geq k\} = ad_\infty.$$

As $\tilde{\varphi}$ is upper semi-continuous, we obtain

$$\begin{aligned}
\lim_{k\to\infty} \tilde{\varphi}(\sup\{&ad_{n-n_\epsilon,m-n_\epsilon} + bd_{n-n_\epsilon,n-n_\epsilon+1} + cd_{m-n_\epsilon,m-n_\epsilon+1} : \\
&n-n_\epsilon, m-n_\epsilon, n-n_\epsilon+1, m-n_\epsilon+1 \geq k\}) \leq \tilde{\varphi}(ad_\infty). \quad (4)
\end{aligned}$$

Clearly,

$$\begin{aligned}
&\sup\{\tilde{\varphi}(ad_{n-n_\epsilon,m-n_\epsilon} + bd_{n-n_\epsilon,n-n_\epsilon+1} + cd_{m-n_\epsilon,m-n_\epsilon+1}) : n-n_\epsilon+1, m-n_\epsilon+1 \geq k\} \\
\leq& \tilde{\varphi}(\sup\{ad_{n-n_\epsilon,m-n_\epsilon} + bd_{n-n_\epsilon,n-n_\epsilon+1} + cd_{m-n_\epsilon,m-n_\epsilon+1} : n-n_\epsilon+1, m-n_\epsilon+1 \geq k\}).
\end{aligned}$$

Thus we have from (4),

$$\begin{aligned}
&\limsup_{k\to\infty} \{\tilde{\varphi}(ad_{n-n_\epsilon,m-n_\epsilon} + bd_{n-n_\epsilon,n-n_\epsilon+1} + cd_{m-n_\epsilon,m-n_\epsilon+1}) : \\
&\qquad m-n_\epsilon+1, n-n_\epsilon+1 \geq k\} \leq \tilde{\varphi}(ad_\infty) \\
&\Rightarrow \limsup_{m,n\to\infty} \{\tilde{\varphi}(ad_{n-n_\epsilon,m-n_\epsilon} + bd_{n-n_\epsilon,n-n_\epsilon+1} + cd_{m-n_\epsilon,m-n_\epsilon+1})\} \leq \tilde{\varphi}(ad_\infty). \quad (5)
\end{aligned}$$

Taking $\limsup$ in (3) and using (5), we have

$$\limsup_{m,n\to\infty} d_{n,m} \leq \tilde{\varphi}(ad_\infty) + \epsilon \;\Rightarrow\; d_\infty \leq \tilde{\varphi}(ad_\infty) + \epsilon$$

which is true for all $\epsilon > 0$ and $a \leq 1$. Hence, we must have $d_\infty = 0$, i.e., $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Therefore, (x_n) is a Cauchy sequence and since (X, d) is a complete metric space, (x_n) converges to an element $p \in X$. Since T is orbitally continuous, it follows that p is a fixed point of the mapping T. □

In the succeeding result, we ascertain the existence of fixed points of Chatterjea type weakly asymptotic contractions. Since the proof is analogous to that of the previous theorem, we omit it.

THEOREM 2.4. *Let (X,d) be a complete metric space and T be a Chatterjea type weakly asymptotic contraction. Further assume that T has a bounded orbit at $x \in X$ and T is orbitally continuous. Then T has a unique fixed point and the sequence $(T^n x)$ converges to that fixed point.*

Now we prove the subsequent fixed point theorem related to a general weak asymptotic contraction.

THEOREM 2.5. *Let (X,d) be a complete metric space, φ be a contractive gauge function, a,b,c be positive real numbers such that $a+b+c=1$. Further, assume that $T : X \to X$ has a bounded orbit at $x \in X$ and T is orbitally continuous. Also assume that T satisfies the following condition: for any $\epsilon > 0$ there exists an $n_\epsilon \in \mathbb{N}$ such that*

$$d(T^{n_\epsilon}x, T^{n_\epsilon}y) \le \varphi(a\max\{d(x,y), d(x,Tx), d(y,Ty)\} + bd(x,Ty) + cd(y,Tx)) + \epsilon$$

holds for all $x,y \in X$. Then T has a unique fixed point in X and sequence $(T^n x)$ converges to that fixed point.

Proof. Let us choose an arbitrary but fixed element $x \in X$ and construct a sequence (x_n) defined by $x_n = T^n x$ for all $n \in \mathbb{N}$. Since T has a bounded orbit, we have $D = \limsup_{n\to\infty} d(x_n, x_{n+1}) < \infty$.

Let $\epsilon > 0$ be arbitrary. Then there exists $n_\epsilon \in \mathbb{N}$ such that for all $x,y \in X$ we have

$$\begin{aligned}
&d(T^{n_\epsilon}x_n, T^{n_\epsilon}x_{n+1})\\
\le&\varphi(a\max\{d(x_n,x_{n+1}), d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\}+bd(x_n,Tx_{n+1})+cd(x_{n+1},Tx_n))+\epsilon\\
\le&\tilde{\varphi}(a\max\{d(x_n,x_{n+1}), d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\}+bd(x_n,x_{n+2})+cd(x_{n+1},x_{n+1}))+\epsilon\\
=&\tilde{\varphi}(a\max\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\}+bd(x_n,x_{n+1})+bd(x_{n+1},x_{n+2}))+\epsilon. \qquad (6)
\end{aligned}$$

Further, we know,

$$\limsup_{n\to\infty}\,[a\max\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\}+bd(x_n,x_{n+1})+bd(x_{n+1},x_{n+2})] = (a+2b)D.$$

Since $\tilde{\varphi}$ is upper semi-continuous,

$$\begin{aligned}
&\limsup_{n\to\infty}\tilde{\varphi}(a\max\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\} + bd(x_n,x_{n+1}) + bd(x_{n+1},x_{n+2}))\\
&\qquad \le \tilde{\varphi}((a+2b)D). \qquad (7)
\end{aligned}$$

Therefore,

$$\limsup_{n\to\infty} d(T^{n_\epsilon}x_n, T^{n_\epsilon}x_{n+1}) \le \tilde{\varphi}((a+2b)D) + \epsilon \Rightarrow D \le \tilde{\varphi}((a+2b)D) + \epsilon. \qquad (8)$$

Interchanging x_n and x_{n+1} in (6), letting $\limsup$ as $n \to \infty$ and using (7), we get

$$D \le \tilde{\varphi}((a+2c)D) + \epsilon. \qquad (9)$$

Adding (8) and (9), we get $2D \le \tilde{\varphi}((a+2b)D) + \tilde{\varphi}((a+2c)D) + 2\epsilon$. If $D \ne 0$, then $2D < (2-a-2c)D + (a+2c)D$ which leads to a contradiction. Hence we have $D = 0$, i.e., $\limsup_{n\to\infty} d(x_n, x_{n+1}) = 0$. Further, we consider $d_{n,m} = d(T^n x, T^m x)$ for all $n, m \geq 1$ and $d_\infty = \limsup_{n,m\to\infty} d_{n,m} = \limsup_{k\to\infty}\{d_{n,m} : n, m \geq k\}$.

Let $\epsilon > 0$ be arbitrary and so there exists an $n_\epsilon \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$

$$\begin{aligned}
d_{n,m} &= d(T^n x, T^m x) = d(T^{n_\epsilon}(T^{n-n_\epsilon}x), T^{n_\epsilon}(T^{m-n_\epsilon}x)) \\
&\leq \varphi(a \max\{d(T^{n-n_\epsilon}x, T^{m-n_\epsilon}x), d(T^{n-n_\epsilon}x, T(T^{n-n_\epsilon}x)), d(T^{m-n_\epsilon}x, T(T^{m-n_\epsilon}x))\} \\
&\quad + bd(T^{n-n_\epsilon}x, T(T^{m-n_\epsilon}x)) + cd(T^{m-n_\epsilon}x, T(T^{n-n_\epsilon}x))) + \epsilon \\
&= \varphi(a \max\{d(x_{n-n_\epsilon}, x_{m-n_\epsilon}), d(x_{n-n_\epsilon}, x_{n-n_\epsilon+1}), d(x_{m-n_\epsilon}, x_{m-n_\epsilon+1})\} \\
&\quad + bd(x_{n-n_\epsilon}, x_{m-n_\epsilon+1}) + cd(x_{m-n_\epsilon}, x_{n-n_\epsilon+1})) + \epsilon \\
&\leq \tilde{\varphi}(a \max\{d(x_{n-n_\epsilon}, x_{m-n_\epsilon}), d(x_{n-n_\epsilon}, x_{n-n_\epsilon+1}), d(x_{m-n_\epsilon}, x_{m-n_\epsilon+1})\} \\
&\quad + bd(x_{n-n_\epsilon}, x_{m-n_\epsilon+1}) + cd(x_{m-n_\epsilon}, x_{n-n_\epsilon+1})) + \epsilon \\
&\leq \tilde{\varphi}(a \max\{d(x_{n-n_\epsilon}, x_{m-n_\epsilon}), d(x_{n-n_\epsilon}, x_{n-n_\epsilon+1}), d(x_{m-n_\epsilon}, x_{m-n_\epsilon+1})\} \\
&\quad + bd(x_{n-n_\epsilon}, x_{n-n_\epsilon+1}) + bd(x_{n-n_\epsilon+1}, x_{m-n_\epsilon+1}) + cd(x_{m-n_\epsilon}, x_{m-n_\epsilon+1}) \\
&\quad + cd(x_{m-n_\epsilon+1}, x_{n-n_\epsilon+1})) + \epsilon
\end{aligned}$$

$$\begin{aligned}
d_{n,m} \leq{} &\tilde{\varphi}(a \max\{d_{n-n_\epsilon,m-n_\epsilon}, d_{n-n_\epsilon,n-n_\epsilon+1}, d_{m-n_\epsilon,m-n_\epsilon+1}\} + bd_{n-n_\epsilon,n-n_\epsilon+1} \\
&+ bd_{n-n_\epsilon+1,m-n_\epsilon+1} + cd_{m-n_\epsilon,m-n_\epsilon+1} + cd_{m-n_\epsilon+1,n-n_\epsilon+1}) + \epsilon. \qquad (10)
\end{aligned}$$

We claim that

$$\begin{aligned}
&\limsup_{n,m\to\infty} \tilde{\varphi}(a \max\{d_{n-n_\epsilon,m-n_\epsilon}, d_{n-n_\epsilon,n-n_\epsilon+1}, d_{m-n_\epsilon,m-n_\epsilon+1}\} + bd_{n-n_\epsilon,n-n_\epsilon+1} \\
&+ bd_{n-n_\epsilon+1,m-n_\epsilon+1} + cd_{m-n_\epsilon,m-n_\epsilon+1} + cd_{m-n_\epsilon+1,n-n_\epsilon+1}) \\
&\leq \tilde{\varphi}(\limsup_{n,m\to\infty}(a \max\{d_{n-n_\epsilon,m-n_\epsilon}, d_{n-n_\epsilon,n-n_\epsilon+1}, d_{m-n_\epsilon,m-n_\epsilon+1}\} + bd_{n-n_\epsilon,n-n_\epsilon+1} \\
&+ bd_{n-n_\epsilon+1,m-n_\epsilon+1} + cd_{m-n_\epsilon,m-n_\epsilon+1} + cd_{m-n_\epsilon+1,n-n_\epsilon+1})).
\end{aligned}$$

Let

$$\begin{aligned}
&\limsup_{n,m\to\infty} \quad (a \max\{d_{n-n_\epsilon,m-n_\epsilon}, d_{n-n_\epsilon,n-n_\epsilon+1}, d_{m-n_\epsilon,m-n_\epsilon+1}\} + bd_{n-n_\epsilon,n-n_\epsilon+1} \\
&+ bd_{n-n_\epsilon+1,m-n_\epsilon+1} + cd_{m-n_\epsilon,m-n_\epsilon+1} + cd_{m-n_\epsilon+1,n-n_\epsilon+1}) = L,
\end{aligned}$$

that is,

$$\begin{aligned}
&\limsup_{k\to\infty} \quad \{a \max\{d_{n-n_\epsilon,m-n_\epsilon}, d_{n-n_\epsilon,n-n_\epsilon+1}, d_{m-n_\epsilon,m-n_\epsilon+1}\} + bd_{n-n_\epsilon,n-n_\epsilon+1} \\
&+ bd_{n-n_\epsilon+1,m-n_\epsilon+1} + cd_{m-n_\epsilon,m-n_\epsilon+1} + cd_{m-n_\epsilon+1,n-n_\epsilon+1} : \\
&m - n_\epsilon + 1, n - n_\epsilon + 1 \geq k\} = L.
\end{aligned}$$

As $\tilde{\varphi}$ is upper semi-continuous, we obtain

$$\begin{aligned}
&\lim_{k\to\infty} \tilde{\varphi}(\sup\{a \max\{d_{n-n_\epsilon,m-n_\epsilon}, d_{n-n_\epsilon,n-n_\epsilon+1}, d_{m-n_\epsilon,m-n_\epsilon+1}\} + bd_{n-n_\epsilon,n-n_\epsilon+1} \\
&+ bd_{n-n_\epsilon+1,m-n_\epsilon+1} + cd_{m-n_\epsilon,m-n_\epsilon+1} + cd_{m-n_\epsilon+1,n-n_\epsilon+1} : \\
&m - n_\epsilon + 1, n - n_\epsilon + 1 \geq k\}) \leq \tilde{\varphi}(L).
\end{aligned}$$

Clearly,

$$\sup\{\tilde{\varphi}(a \max\{d_{n-n_\epsilon,m-n_\epsilon}, d_{n-n_\epsilon,n-n_\epsilon+1}, d_{m-n_\epsilon,m-n_\epsilon+1}\} + bd_{n-n_\epsilon,n-n_\epsilon+1}$$

$$
\begin{aligned}
&+ bd_{n-n_\epsilon+1,m-n_\epsilon+1} + cd_{m-n_\epsilon,m-n_\epsilon+1} + cd_{m-n_\epsilon+1,n-n_\epsilon+1}) :\\
&m - n_\epsilon + 1, n - n_\epsilon + 1 \geq k\}\\
&\leq \tilde{\varphi}(\sup\{a \max\{d_{n-n_\epsilon,m-n_\epsilon}, d_{n-n_\epsilon,n-n_\epsilon+1}, d_{m-n_\epsilon,m-n_\epsilon+1}\} + bd_{n-n_\epsilon,n-n_\epsilon+1}\\
&+ bd_{n-n_\epsilon+1,m-n_\epsilon+1} + cd_{m-n_\epsilon,m-n_\epsilon+1} + cd_{m-n_\epsilon+1,n-n_\epsilon+1} :\\
&m - n_\epsilon + 1, n - n_\epsilon + 1 \geq k\}).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\limsup_{k\to\infty}\{\tilde{\varphi}(a \max\{d_{n-n_\epsilon,m-n_\epsilon}, d_{n-n_\epsilon,n-n_\epsilon+1}, d_{m-n_\epsilon,m-n_\epsilon+1}\} + bd_{n-n_\epsilon,n-n_\epsilon+1}\\
&+ bd_{n-n_\epsilon+1,m-n_\epsilon+1} + cd_{m-n_\epsilon,m-n_\epsilon+1} + cd_{m-n_\epsilon+1,n-n_\epsilon+1}) :\\
&m - n_\epsilon + 1, n - n_\epsilon + 1 \geq k\} \leq \tilde{\varphi}(L)\\
\Rightarrow &\limsup_{m,n\to\infty}\{\tilde{\varphi}(a \max\{d_{n-n_\epsilon,m-n_\epsilon}, d_{n-n_\epsilon,n-n_\epsilon+1}, d_{m-n_\epsilon,m-n_\epsilon+1}\} + bd_{n-n_\epsilon,n-n_\epsilon+1}\\
&+ bd_{n-n_\epsilon+1,m-n_\epsilon+1} + cd_{m-n_\epsilon,m-n_\epsilon+1} + cd_{m-n_\epsilon+1,n-n_\epsilon+1})\}\\
&\leq \tilde{\varphi}(\limsup_{m,n\to\infty}\{a \max\{d_{n-n_\epsilon,m-n_\epsilon}, d_{n-n_\epsilon,n-n_\epsilon+1}, d_{m-n_\epsilon,m-n_\epsilon+1}\} + bd_{n-n_\epsilon,n-n_\epsilon+1}\\
&+ bd_{n-n_\epsilon+1,m-n_\epsilon+1} + cd_{m-n_\epsilon,m-n_\epsilon+1} + cd_{m-n_\epsilon+1,n-n_\epsilon+1}\})\\
&= \tilde{\varphi}(ad_\infty + bd_\infty + cd_\infty). \qquad (11)
\end{aligned}
$$

From (10) and (11), we have

$$
\limsup_{m,n\to\infty} d_{n,m} \leq \tilde{\varphi}((a+b+c)d_\infty) + \epsilon \quad \Rightarrow d_\infty \leq \tilde{\varphi}((a+b+c)d_\infty) + \epsilon
$$

which is true for all $\epsilon > 0$, we must have $d_\infty = 0$, i.e., $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Therefore, (x_n) is a Cauchy sequence and since (X, d) is a complete metric space, (x_n) converges to an element $p \in X$. Since T is orbitally continuous, it follows that p is a fixed point of the mapping T. □

We authenticate our findings by the following numerical examples.

Example 2.6. We consider the set $X = [0, 1]$ equipped with the usual metric d, and also, define a self-map T on X by

$$
Tx = \begin{cases} \frac{(x+\frac{1}{4})}{3}, & \text{if } 0 \leq x < \frac{1}{2};\\ \frac{x}{2}, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}
$$

Moreover, we consider $\varphi(t) = \frac{t}{4}$ for all $t \in [0, \infty)$. Now we discuss the following three possible cases.

Case-I: $x \in [0, \frac{1}{2})$ and $y \in [\frac{1}{2}, 1]$.

Since $\frac{1}{2^n} \to 0$ as $n \to \infty$, for given $\epsilon = \frac{1}{2}$, there exists $m \in \mathbb{N}$ such that

$$
\frac{1}{2^n} < \frac{1}{2} \Rightarrow \frac{y}{2^n} \leq \frac{1}{2^n} < \frac{1}{2} \quad \text{for all } n \geq m.
$$

Therefore, for all $n \geq m$, we have:

$$
T^n y = \frac{\frac{y}{2^m} + \frac{1}{4}\left(\frac{3^{n-m}-1}{2}\right)}{3^{n-m}} = \frac{y + (3^{n-m}-1)\frac{2^{m-1}}{4}}{2^m.3^{n-m}} = \frac{y}{2^m.3^{n-m}} + \frac{(3^{n-m}-1)\frac{2^{m-1}}{4}}{2^m.3^{n-m}}
$$

$$=\frac{y+\frac{1}{4}}{2^m.3^{n-m}}+\frac{(2^{m-1}.3^{n-m}-2^{m-1}-1)}{2^{m+2}.3^{n-m}}=\frac{y+\frac{1}{4}}{2^m.3^{n-m}}+\frac{1}{8}-\left(\frac{1}{8.3^{n-m}}+\frac{1}{2^{m+2}.3^{n-m}}\right).$$

Now we obtain,

$$|T^nx-T^ny|=\left|\frac{x+\frac{1}{4}\frac{(3^n-1)}{2}}{3^n}-\frac{y+\frac{(3^{n-m}-1)2^m}{2.4}}{2^m.3^{n-m}}\right|=\left|\frac{x+\frac{1}{4}}{3^n}-\frac{y}{2^m.3^{n-m}}+\frac{3^n-3}{8.3^n}-\frac{3^{n-m}-1}{8.3^{n-m}}\right|$$

$$=\left|\frac{x+\frac{1}{4}}{3^n}-\frac{y}{2^m.3^{n-m}}+\frac{1}{8}\left(\frac{1}{3^{n-m}}-\frac{1}{3^{n-1}}\right)\right|\leq\left|\frac{x+\frac{1}{4}}{3^n}-\frac{y}{2^m.3^{n-m}}\right|+\frac{1}{8}\left|\frac{1}{3^{n-m}}-\frac{1}{3^{n-1}}\right|$$

$$\leq\left|\frac{x}{3^{n-1}}-\frac{y}{2^{m-1}.3^{n-m}}\right|+\left|\frac{x}{3^{n-1}}-\frac{x+\frac{1}{4}}{3^n}\right|+\left|\frac{y}{2^{m-1}.3^{n-m}}-\frac{y}{2^m.3^{n-m}}\right|+\frac{1}{8}\left|\frac{1}{3^{n-m}}-\frac{1}{3^{n-1}}\right|$$

$$\leq\left|\frac{x}{3^{n-1}}-\frac{y}{3^{n-1}}\right|+\left|\frac{x}{3^{n-1}}-\frac{x+\frac{1}{4}}{3^n}\right|+\left|\frac{y}{2^{m-1}.3^{n-m}}-\frac{y}{2^m.3^{n-m}}\right|$$

$$+|y|\left|\frac{1}{3^{n-1}}-\frac{1}{2^{m-1}.3^{n-m}}\right|+\frac{1}{8}\left|\frac{1}{3^{n-m}}-\frac{1}{3^{n-1}}\right|$$

$$\leq\frac{d(x,y)}{3^{n-1}}+\frac{d(x,Tx)}{3^{n-1}}+\frac{d(y,Ty)}{2^{m-1}.3^{n-m}}+\left|\frac{1}{3^{n-1}}-\frac{1}{2^{m-1}.3^{n-m}}\right|+\frac{1}{8}\left|\frac{1}{3^{n-m}}-\frac{1}{3^{n-1}}\right|. \tag{12}$$

Let $x_n=\left|\frac{1}{3^{n-1}}-\frac{1}{2^{m-1}.3^{n-m}}\right|+\frac{1}{8}\left|\frac{1}{3^{n-m}}-\frac{1}{3^{n-1}}\right|$. Since $x_n\to 0$ as $n\to\infty$, for a given $\epsilon>0$ there is $k\in\mathbb{N}$, such that $|x_n|<\epsilon$ for all $n\geq k$. Let us consider $k'=\max\{k,m+3\}$. Hence from (12), for all $n\geq k'$, we get

$$\begin{aligned}d(T^nx,T^ny)&\leq\frac{1}{12}d(x,y)+\frac{1}{12}d(x,Tx)+\frac{1}{12}d(y,Ty)+\epsilon\\&\leq\varphi\left(\frac{1}{3}d(x,y)+\frac{1}{3}d(x,Tx)+\frac{1}{3}d(y,Ty)\right)+\epsilon.\end{aligned} \tag{13}$$

Case-II: $x,y\in[0,\frac{1}{2})$.

For this case, we obtain,

$$|T^nx-T^ny|=\left|\frac{x+\frac{1}{4}\frac{(3^n-1)}{2}}{3^n}-\frac{y+\frac{1}{4}\left(\frac{3^n-1}{2}\right)}{3^n}\right|=\left|\frac{x+\frac{1}{4}}{3^n}-\frac{y+\frac{1}{4}}{3^n}\right|$$

$$\leq\left|\frac{x}{3^{n-1}}-\frac{y}{3^{n-1}}\right|+\left|\frac{x}{3^{n-1}}-\frac{x+\frac{1}{4}}{3^n}\right|+\left|\frac{y}{3^{n-1}}-\frac{y+\frac{1}{4}}{3^n}\right|$$

$$\leq\frac{1}{3^{n-1}}\left\{|x-y|+\left|x-\frac{x+\frac{1}{4}}{3}\right|+\left|y-\frac{y+\frac{1}{4}}{3}\right|\right\}$$

$$\leq\frac{1}{12}(d(x,y)+d(x,Tx)+d(y,Ty))\leq\varphi\left(\frac{1}{3}d(x,y)+\frac{1}{3}d(x,Tx)+\frac{1}{3}d(y,Ty)\right)+\epsilon$$

for all $n\geq 4$ and $\epsilon>0$ and hence, for all $n\geq 4$,

$$\begin{aligned}d(T^nx,T^ny)&\leq\frac{1}{12}d(x,y)+\frac{1}{12}d(x,Tx)+\frac{1}{12}d(y,Ty)+\epsilon\\&\leq\varphi\left(\frac{1}{3}d(x,y)+\frac{1}{3}d(x,Tx)+\frac{1}{3}d(y,Ty)\right)+\epsilon.\end{aligned} \tag{14}$$

Case-III: $x, y \in [\frac{1}{2}, 1]$.

In this case, we have,

$$|T^n x - T^n y| = \left|\frac{x}{2^m.3^{n-m}} - \frac{y}{2^m.3^{n-m}}\right|$$
$$\leq \left|\frac{x}{3^{n-m}.2^{m-1}} - \frac{y}{3^{n-m}.2^{m-1}}\right| + \left|\frac{x}{3^{n-m}.2^{m-1}} - \frac{x}{3^{n-m}.2^{m}}\right| + \left|\frac{y}{3^{n-m}.2^{m-1}} - \frac{y}{3^{n-m}.2^{m}}\right|$$
$$\leq \left|\frac{x}{3^{n-m}.2^{m-1}} - \frac{y}{3^{n-m}.2^{m-1}}\right| + \frac{1}{3^{n-m}.2^{m-1}} d(x, Tx) + \frac{1}{3^{n-m}.2^{n-m}} d(y, Ty)$$
$$\leq \frac{1}{3^{n-m}.2^{m-1}} \left(d(x, y) + d(x, Tx) + d(y, Ty)\right) \leq \frac{1}{12}(d(x, y) + d(x, Tx) + d(y, Ty))$$
$$\leq \varphi\left(\frac{1}{3}d(x, y) + \frac{1}{3}d(x, Tx) + \frac{1}{3}d(y, Ty)\right) + \epsilon$$

for all $n \geq 4$ and $\epsilon > 0$. Let $n_\epsilon = \max\{k', 4\}$. Then,

$$\begin{aligned} d(T^n x, T^n y) &\leq \frac{1}{12}d(x, y) + \frac{1}{12}d(x, Tx) + \frac{1}{12}d(y, Ty) + \epsilon \\ &\leq \varphi\left(\frac{1}{3}d(x, y) + \frac{1}{3}d(x, Tx) + \frac{1}{3}d(y, Ty)\right) + \epsilon \end{aligned} \tag{15}$$

for all $n \geq 4$. Hence combining (13), (14) and (15), we get

$$d(T^{n_\epsilon} x, T^{n_\epsilon} y) \leq \varphi\left(\frac{1}{3}d(x, y) + \frac{1}{3}d(x, Tx) + \frac{1}{3}d(y, Ty)\right) + \epsilon$$

for all $x, y \in X$ and therefore T is a Reich type weakly asymptotic contraction on X. Employing Theorem 2.3, we can conclude that T has a unique fixed point and here it is $x = \frac{1}{8}$.

Example 2.7. Let $([0, 1] \times [0, 1], d)$ be a metric space where the metric is defined as $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ and consider the map $T : [0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ defined by

$$T(x, y) = \begin{cases} \left(\frac{x^2}{2}, \frac{y}{8}\right), & \text{if } (x, y) \in [0, \frac{1}{2}) \times [0, 1]; \\ \left(\frac{x}{8}, \frac{y^2}{2}\right), & \text{if } (x, y) \in [\frac{1}{2}, 1] \times [0, 1]. \end{cases}$$

We proceed further as in Example 2.6 and study all possible cases. Summing up those, we can conclude that T satisfies the assumptions of Theorem 2.3 and owns a unique fixed point.

3. Solutions to Cantilever Beam problems

In this section, we analyse the solvability of the following fourth-order two-point boundary value problem

$$\left.\begin{aligned} u^{(4)}(t) = K(t, u(t)),\ a < t < b; \\ u(a) = u'(a) = u''(b) = u'''(b) = 0 \end{aligned}\right\} \tag{16}$$

where $K : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The above boundary value problem is a particular example of beam problem when uniform load is distributed, that is, the above boundary value problem is a cantilever beam problem. Equation (16) is equivalent to the following integral equation:

$$u(t) = \int_a^b G(t,x)K(x,u(x))dx, \text{ for } t \in [a,b],$$

where $G : [a,b] \times [a,b] \to \mathbb{R}$ is the Green's function, defined by

$$G(t,x) = \begin{cases} \frac{1}{2}(t-a)^2(x-a) - \frac{1}{6}(t-a)^3 + \frac{1}{6}(t-x)^3, & \text{if } a \le x \le t \le b; \\ \frac{1}{2}(t-a)^2(x-a) - \frac{1}{6}(t-a)^3, & \text{if } a \le t \le x \le b. \end{cases}$$

Now we prove the following theorem concerning the existence of unique solution of the boundary value problem (16).

Theorem 3.1. *Suppose that for any $u, v \in C[a,b]$, the following holds:*

$$\sup_{x\in[a,b]} \big|K(x,u(x)) - K(x,v(x))\big|$$
$$\le \max\left\{ \sup_{x\in[a,b]} |u(x)-v(x)|, \sup_{t\in[a,b]} \left|u(t) - \int_a^b G(t,x)K(x,u(x))dx\right|, \right.$$
$$\left. \sup_{t\in[a,b]} \left|v(t) - \int_a^b G(t,x)K(x,v(x))dx\right| \right\}.$$

Then the boundary value problem (16) *has a unique solution in* $C[a,b]$ *if the function* K *is bounded and* $\sup_{t\in[a,b]} \int_a^b G(t,x)dx < 1$.

Proof. Let us consider the complete metric space $(C[a,b], d)$ where d is the sup metric on $C[a,b]$. First, we define a mapping T on $C[a,b]$ by

$$(Tu)(t) = \int_a^b G(t,x)K(x,u(x))dx, \text{ for all } u \in C[a,b] \text{ and } t \in [a,b].$$

Then it is easy to note that T is a self-map on $C[a,b]$ and T is continuous. Again since K is bounded, T has bounded orbits at all $u \in C[a,b]$.

Now for any $u, v \in C[a,b]$ and for any $t \in [a,b]$, we have

$$|(Tu)(t) - (Tv)(t)| = \left| \int_a^b G(t,x)K(x,u(x))dx - \int_a^b G(t,x)K(x,v(x))dx \right|$$
$$= \left| \int_a^b G(t,x)\left\{K(x,u(x)) - K(x,v(x))\right\}dx \right| \le \int_a^b G(t,x)\left|K(x,u(x)) - K(x,v(x))\right|dx$$
$$\le \sup_{x\in[a,b]} |K(x,u(x)) - K(x,v(x))| \int_a^b G(t,x)dx. \tag{17}$$

If $\max\left\{ \sup_{x\in[a,b]} |u(x)-v(x)|, \sup_{t\in[a,b]} \left|u(t) - \int_a^b G(t,x)K(x,u(x))dx\right|, \right.$

$\sup_{t\in[a,b]}\left|v(t)-\int_a^b G(t,x)K(x,v(x))dx\right|\Bigg\} = \sup_{x\in[a,b]}|u(x)-v(x)|$, then using given condition and (17), we get

$$|(Tu)(t)-(Tv)(t)| \le \sup_{x\in[a,b]}|u(x)-v(x)|\int_a^b G(t,x)dx = d(u,v)\int_a^b G(t,x)dx. \quad (18)$$

If $\max\Bigg\{\sup_{x\in[a,b]}|u(x)-v(x)|, \sup_{t\in[a,b]}\left|u(t)-\int_a^b G(t,x)K(x,u(x))dx\right|,$

$\sup_{t\in[a,b]}\left|v(t)-\int_a^b G(t,x)K(x,v(x))dx\right|\Bigg\} = \sup_{t\in[a,b]}\left|u(t)-\int_a^b G(t,x)K(x,u(x))dx\right|,$

then using given condition and (17), we get

$$|(Tu)(t)-(Tv)(t)| \le \sup_{t\in[a,b]}\left|u(t)-\int_a^b G(t,x)K(x,u(x))dx\right|\int_a^b G(t,x)dx$$

$$= \sup_{t\in[a,b]}|u(t)-(Tu)(t)|\int_a^b G(t,x)dx = d(u,Tu)\int_a^b G(t,x)dx. \quad (19)$$

Similarly if $\max\Bigg\{\sup_{x\in[a,b]}|u(x)-v(x)|, \sup_{t\in[a,b]}\left|u(t)-\int_a^b G(t,x)K(x,u(x))dx\right|,$

$\sup_{t\in[a,b]}\left|v(t)-\int_a^b G(t,x)K(x,v(x))dx\right|\Bigg\} = \sup_{t\in[a,b]}\left|v(t)-\int_a^b G(t,x)K(x,v(x))dx\right|,$

then we have

$$|(Tu)(t)-(Tv)(t)| \le d(v,Tv)\int_a^b G(t,x)dx. \quad (20)$$

Thus using (18)-(20) in (17), we get

$$|(Tu)(t)-(Tv)(t)| \le \max\big\{d(u,v), d(u,Tu), d(v,Tv)\big\}\int_a^b G(t,x)dx, \quad \text{for all } t\in[a,b]$$

$$\Rightarrow \sup_{t\in[a,b]}|(Tu)(t)-(Tv)(t)| \le \max\big\{d(u,v), d(u,Tu), d(v,Tv)\big\}\sup_{t\in[a,b]}\int_a^b G(t,x)dx$$

$$\Rightarrow d(Tu,Tv) \le \alpha\max\big\{d(u,v), d(u,Tu), d(v,Tv)\big\}, \quad \text{where } \alpha = \sup_{t\in[a,b]}\int_a^b G(t,x)dx.$$

Then by using stated condition, we have $0\le\alpha<1$. Let us choose $\varphi\in\Phi$ defined by $\varphi(t)=\alpha t$ for all $t\in[a,b]$. Then for any given $\epsilon>0$, if we choose $n_\epsilon=1$, we have $d(T^{n_\epsilon}u, T^{n_\epsilon}v) \le \varphi(a_1\max\{d(u,v), d(u,Tv), d(v,Tv)\} + b_1 d(u,Tv) + c_1 d(v,Tu)) + \epsilon$, for all $u,v\in C[a,b]$, where $a_1=1$, $b_1=c_1=0$. Hence by Theorem 2.5, T has a unique fixed point in $C[a,b]$, that is, the boundary value problem (16) has a unique solution in $C[a,b]$. □

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