MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
75, 2 (2023), 97–105
June 2023

research paper
оригинални научни рад
DOI: 10.57016/MV-5375fdsg

CHARACTERIZATION OF MATRICES OF OPERATORS ON THE L_ϕ-VALUED CESÀRO SEQUENCE SPACES

Haryadi, Supama and Atok Zulijanto

Abstract. In this paper, we characterized the matrices of operators that transform the generalized Cesàro sequence space to the convergence sequence space in Banach spaces. Our results generalize the characterization of the sequence space in L_p, $1 < p < \infty$.

1. Introduction

Let X and Y be Banach spaces and $A = (A_{nk})$ be an infinite matrix of operators from X into Y. The investigation of characterizations of the class of matrices of operators that transform an X-valued sequence space into a Y-valued sequence space has been carried out by several authors (e.g., [6, 7]). Başar et al. in [1] investigated matrices transformation on some sequence spaces related to strong Cesàro summability. Related to the characterization of matrices of operators, Yilmaz and Ozdemir [13] examined the Köthe-Toeplitz duals of some vector-valued Orlicz sequence spaces. The Köthe-Toeplitz duals for the sequences in a generalized Orlicz space are examined in [4]. In [11], Malkowsky and Veličković determined β-duals on some new sequence spaces. The duals and matrices transformation on some spaces related to Cesàro sequence space were also discussed in [10].

Given the generalized Orlicz space L_ϕ associated with an Orlicz function ϕ, the space of all sequences in L_ϕ is denoted by $\omega(L_\phi)$. For any $u \in L_\phi$ and $m = 1, 2, \ldots$, let $e^m u$ denote a sequence in L_ϕ with $e^m_k = u$ if $k = m$ and $e^m_k = \theta$ for $k \neq m$. Here, θ denotes the zero vector in L_ϕ. If $v = (v_k)$ is a member of $\omega(L_\phi)$, then the notation $\sum_k^\infty A_{nk} v_k$ is also written as $A_n v$.

Let ϕ be an Orlicz function that satisfies the Δ_2-condition. We define the following spaces:

$$W_{0,\phi} = \Big\{ (u_k) \in \omega(L_\phi) : \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \int_E \phi(u_k(x)) = 0 \Big\},$$

2020 Mathematics Subject Classification: 40A30, 40J05

Keywords and phrases: Matrices of operators; Cesàro sequence spaces; Orlicz function.

$$W_\phi = \{(u_k) \in \omega(L_\phi) : (u_k - u_0) \in W_{0,\phi} \text{ for some } u_0 \in L_\phi\},$$

$$W_{\infty,\phi} = \left\{(u_k) \in \omega(L_\phi) : \sup_{N\in\mathbb{N}} \frac{1}{N} \sum_{k=1}^{N} \int_E \phi(u_k(x)) < \infty\right\}.$$

Let $c(Y)$ denote the space of all convergent sequences in a Banach space Y and let $W \in \{W_{0,\phi}, W_\phi, W_{\infty,\phi}\}$. In this paper, we give the characterization of the class $(W, c(Y))$, i.e. the class of all matrices that transform the spaces W to the space $c(Y)$. Based on the characterization, we then determine the class $(W, c(Y))$ for $\phi = |\cdot|^p$, $1 < p < \infty$.

2. Preliminaries

The symbols $\mathbb{N}$ and $\mathbb{R}$ denote the set of all natural numbers and the set of all real numbers, respectively. Let $\phi : \mathbb{R} \to [0,\infty)$ be an Orlicz function, that is, ϕ is even, continuous, convex, $\phi(x) = 0$ if and only if $x = 0$, and $\lim_{x\to\infty} \phi(x) = \infty$. The complementary to the Orlicz function ϕ is a function ψ such that $|xy| \le \phi(x) + \psi(y)$, for every $x, y \in \mathbb{R}$. For any Orlicz function ϕ, the function ψ defined by $\psi(y) = \sup\{|y|x - \phi(x) : x \ge 0\}$ is an Orlicz function complementary to ϕ. An Orlicz function ϕ is said to satisfy the Δ_2-condition if there is a $K > 0$ such that $\phi(2x) \le K\phi(x)$ for each $x \ge 0$ (see e.g., [5]). We denote by ϕ^{-1} the inverse function of the Orlicz function ϕ in the non-negative values argument.

Let (ϕ, ψ) be a pair of complementary Orlicz functions and E a bounded closed subset of $\mathbb{R}$. We denote by L_ϕ the space of all Lebesgue measurable real valued functions u on E such that $|\int_E u(x)v(x)\,dx| < \infty$ for every v with $\int_E \psi(v(x))\,dx < \infty$. Furthermore, L_ϕ is a Banach space with respect to the Orlicz norm

$$\|u\|_\phi = \sup\left\{\left|\int_E u(x)g(x)\,dx\right| : \rho_\psi(u) \le 1\right\},$$

where $\rho_\psi(u) = \int_E \psi(u(x))\,dx$ (see, e.g., [5, 12]).

By the Luxemburg norm on L_ϕ we mean a function $\|\cdot\|_{(\phi)}$ on L_ϕ such that $\|u\|_{(\phi)} = \inf\{t > 0 : \rho_\phi(u/t) \le 1\}$. It can be shown that $\|u\|_{(\phi)} \le \|u\|_\phi \le 2\|u\|_{(\phi)}$ for all $u \in L_\phi$, that is $\|\cdot\|_\phi$ and $\|\cdot\|_{(\phi)}$ are equivalent. Furthermore, if the Orlicz function ϕ satisfies the Δ_2-condition, then $\rho_\phi(u/\|u\|_{(\phi)}) = 1$ (see, e.g., [5]).

If the Orlicz function ϕ satisfies the Δ_2-condition, it can be shown that there exists a constant $c > 0$ such that

$$\int_E \phi(au(x))\,dx \le c\phi(a), \quad \text{for each} \quad a > 0 \quad \text{and} \quad \|u\|_\phi \le 1 \tag{1}$$

(see [4]). In [3], we showed that the space $W_{\infty,\phi}$ is complete with respect to the Luxemburg norm

$$\|u\| = \inf\left\{t > 0 : \sup_{N\in\mathbb{N}} \frac{1}{N} \sum_{k=1}^{N} \int_E \phi\left(\frac{u_k(x)}{t}\right) \le 1\right\}.$$

Furthermore, W_ϕ is a closed subspace of $W_{\infty,\phi}$.

Let $\mathcal{B}(L_\phi, Y)$ be the collection of all bounded linear mappings from L_ϕ into Y and Y^* denote the continuous dual of Y. We also write $\langle f, y\rangle = f(y)$ for $f \in Y^*$ and $y \in Y$. The adjoint of $T \in \mathcal{B}(L_\phi, Y)$ is an operator $T^* \in \mathcal{B}(Y^*, L^*_\phi)$ such that $\langle f, Tu\rangle = \langle T^* f, u\rangle$ for all $f \in Y^*$ and $u \in L_\phi$. The notations U and U^* represent the sets $\{u \in L_\phi : \|u\|_\phi \leq 1\}$ and $\{f \in Y^* : \|f\| \leq 1\}$, respectively. Recall that for every $y \in Y$, $y \neq \theta$, there exists $f \in U^*$ such that $\|y\|_Y = f(y)$.

Let (T_k) be a sequence in $\mathcal{B}(L_\phi, Y)$ and $f \in U^*$. Since for each k, $T^*_k f \in L^*_\phi$, then there exists $u_k \in U$ such that

$$\|T^*_k f\| \leq 2|T^*_k f u_k|. \tag{2}$$

Let (a_k) be a sequence of real numbers and u_k as is (2) for each $k \in \mathbb{N}$. It is clear that the sequence (v_k) with

$$v_k = |a_k| \operatorname{sgn}(T^*_k f u_k) u_k \tag{3}$$

is a sequence in L_ϕ.

On L_p, $1 \leq p < \infty$, by taking $\phi(x) = |x|^p$, it is well known that $\phi(\|u\|_{(\phi)}) = \rho_\phi(u)$, but in the Orlicz space, this property is not always satisfied even if the Orlicz function ϕ satisfies the Δ_2-condition (see [2]). However, if we restrict on the terms of $W_{\infty,\phi}$, we have the following result.

Lemma 2.1 ([4]). *If an Orlicz function ϕ satisfies the Δ_2-condition and $(u_k) \in W_{\infty,\phi}$, then there exists $c > 0$ such that $\phi(\|u_k\|_\phi) \leq c\rho_\phi(u_k)$, $\quad \forall k \in \mathbb{N}$.*

Let ω denote the space of all sequences in $\mathbb{R}$ and let the Orlicz function ϕ satisfy the Δ_2-condition. To discuss our results, we will need the following spaces.

$$w_{0,\phi} = \left\{(a_k) \in \omega : \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \phi(a_k) = 0\right\},$$

$$w_\phi = \{(a_k) \in \omega : (a_k - a_0) \in w_{0,\phi} \text{ for some } a_0 \in \mathbb{R}\},$$

$$w_{\infty,\phi} = \left\{(a_k) \in \omega : \sup_{N\in\mathbb{N}} \frac{1}{N} \sum_{k=1}^{N} \phi(a_k) < \infty\right\}.$$

In the case of $\phi(x) = |x|^p$, $1 \leq p < \infty$, we write $w_{0,\phi} = w_{0,p}$ and $w_{\infty,\phi} = w_{\infty,p}$.

Let (a_k) be a sequence of real numbers. The notation $\sum_r a_k$ stands for the sum of all a_k with $2^r \leq k < 2^{r+1}$. For our discussion that follows, we rewrite the results from [9] in the following version.

Lemma 2.2 ([9]). *Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and (b_k) be a sequence of real numbers. The following statements are equivalent.*

(i) $\sum_{k=1}^{\infty} |b_k||a_k| < \infty$, *for each* $(a_k) \in w_{0,p}$.

(ii) $\sum_{k=1}^{\infty} |b_k||a_k| < \infty$, *for each* $(a_k) \in w_{\infty,p}$.

(iii) $\sum_{r=0}^{\infty} 2^{r/p} \left(\sum_r |b_k|^q\right)^{1/q} < \infty$.

3. Main results

We begin this section by proving the following three theorems, that will be useful for proving our main results.

THEOREM 3.1. *Let ϕ be an Orlicz function that satisfies the Δ_2-condition and $A_k \in \mathcal{B}(L_\phi, Y)$ for each $k \in \mathbb{N}$. The following statements are equivalent.*
(a) $\sum_{k=1}^{\infty} A_k u_k$ is convergent for each $(u_k) \in W_{0,\phi}$.

(b) $\sum_{k=1}^{\infty} |a_k| \|A_k^* f\| < \infty$ for each $(a_k) \in w_{0,\phi}$ and for each $f \in U^*$.

Proof. Assume (a) is true and let $(a_k) \in w_{0,\phi}$ and $f \in U^*$. Define the sequence (v_k) where v_k is as in (3) with A_k^* in the place of T_k^*. Following (1), there exists $c > 0$ such that

$$\frac{1}{N}\sum_{k=1}^{N}\int_E \phi(v_k(x))\,dx = \frac{1}{N}\sum_{k=1}^{N}\int_E \phi(a_k u_k(x))\,dx \le \frac{c}{N}\sum_{k=1}^{N}\phi(a_k) \to 0,$$

whenever $N \to \infty$, i.e. $(v_k) \in W_{0,\phi}$. Hence, $\sum_{k=1}^{\infty} A_k v_k$ is convergent, and it implies that $\sum_{k=1}^{\infty} A_k^* f v_k$ is convergent. Since $\sum_{k=1}^{\infty} A_k^* f v_k = \sum_{k=1}^{\infty} |a_k||A_k^* f u_k|$, then we have

$$\sum_{k=1}^{\infty} |a_k| \|A_k^* f\| \le 2\sum_{k=1}^{\infty} |a_k||A_k^* f u_k| < \infty.$$

Thus, (a) implies (b).

Assume (b) is true and take any $(u_k) \in W_{0,\phi}$. By Lemma 2.1, it is easy to see that $(\|u_k\|_\phi)_{k\in\mathbb{N}} \in w_{0,\phi}$, and hence $\sum_{k=1}^{\infty} \|A_k^* f\| \|u_k\|_\phi < \infty$. Since

$$\begin{aligned}\|\sum_{k=m}^{n} A_k u_k\|_Y &= \left|\langle f, \sum_{k=m}^{n} A_k u_k\rangle\right|, \quad \text{for some } f \in U^* \\ &\le \sum_{k=m}^{n} |\langle f, A_k u_k\rangle| = \sum_{k=m}^{n} |A_k^* f u_k| \le \sum_{k=m}^{n} \|A_k^* f\| \|u_k\|_\phi,\end{aligned}$$

then $\|\sum_{k=m}^{n} A_k u_k\|_Y \to 0$ as $m, n \to \infty$. By the completeness of Y, $\sum_{k=1}^{\infty} A_k u_k$ is convergent. Hence, (b) implies (a). □

THEOREM 3.2. *Let ϕ be an Orlicz function that satisfies the Δ_2-condition and $A_k \in \mathcal{B}(L_\phi, Y)$ for each $k \in \mathbb{N}$. The following statements are equivalent.*
(a) $\sum_{k=1}^{\infty} A_k u_k$ is convergent for each $(u_k) \in W_{0,\phi}$.

(b) $\sum_{k=1}^{\infty} A_k u_k$ is convergent for each $(u_k) \in W_\phi$.

Proof. Since $W_{0,\phi} \subset W_\phi$, then (b) implies (a).

For the converse, assume that (a) is true and let $(u_k) \in W_\phi$. Let $u_0 \in L_\phi$ such that $(u_k - u_0) \in W_{0,\phi}$, then $\sum_{k=1}^{\infty} A_k(u_k - u_0)$ is convergent. Since the function ϕ satisfies the Δ_2-condition, then there exists $p > 1$ such that $\phi(x) \le c|x|^p$, for some $c > 0$ [12, Corollary 5, Chap. II]. It implies $w_{0,p} \subset w_{0,\phi}$. Let $(a_k) \in w_{0,p}$.

By Theorem 3.1, $\sum_{k=1}^{\infty}|a_k|\|A_k^*f\| < \infty$. Hence, $\sum_{r=0}^{\infty}2^{r/p}(\sum_r\|A_k^*f\|^q)^{1/q} < \infty$, $1/p+1/q=1$ (see [8]). For any $\varepsilon>0$, let $r_0\in\mathbb{N}$ be such that

$$\sum_{r=r_0}^{\infty}2^{r/p}\left(\sum_r\|A_k^*f\|^q\right)^{1/q} < \frac{\varepsilon}{\|u_0\|_\phi+1}.$$

For each $m,n\geq 2^{r_0}$,

$$\begin{aligned}
&\|\sum_{k=m}^{n}A_ku_0\|_Y = |\langle f,\sum_{k=m}^{n}A_ku_0\rangle|, \quad \text{for some } f\in U^*\\
&\leq \sum_{k=m}^{n}|A_k^*fu_0| \leq \sum_{k=m}^{n}\|A_k^*f\|\|u_0\|_\phi \leq \sum_{r=r_0}^{\infty}2^{r/p}\left(\sum_r\|A_k^*f\|^q\right)^{1/q}\left(\frac{1}{2^r}\sum_r\|u_0\|_\phi^p\right)^{1/p}\\
&\leq \sum_{r=r_0}^{\infty}2^{r/p}\left(\sum_r\|A_k^*f\|^q\right)^{1/q}\|u_0\|_\phi < \varepsilon.
\end{aligned}$$

Since Y is complete, then $\sum_{k=1}^{\infty}A_ku_0$ is convergent. Therefore

$$\sum_{k=1}^{\infty}A_ku_k = \sum_{k=1}^{\infty}A_k(u_k-u_0)+\sum_{k=1}^{\infty}A_ku_0$$

is convergent. Thus, (a) implies (b). □

THEOREM 3.3. *Let ϕ be an Orlicz function that satisfies the Δ_2-condition and $A_k\in\mathcal{B}(L_\phi,Y)$ for each $k\in\mathbb{N}$. The following statements are equivalent.*
(a) $\sum_{k=1}^{\infty}A_ku_k$ is convergent for each $(u_k)\in W_{\infty,\phi}$.
(b) $\sum_{k=1}^{\infty}|a_k|\|A_k^*f\|<\infty$ for each $(a_k)\in w_{\infty,\phi}$.

Proof. The proof that (a) implies (b) is analogous to the proof of Theorem 3.1, by replacing $(a_k)\in w_{\infty,\phi}$ in the proof of Theorem 3.1.

For the converse, take any $(u_k)\in W_{\infty,\phi}$. Since $(\|u_k\|)\in w_{\infty,\phi}$, then by (b) we have $\sum_{k=1}^{\infty}\|A_k^*f\|\|u_k\|<\infty$. Since Y is complete, then $\sum_{k=1}^{\infty}A_ku_k$ is convergent. □

Now, we are in the position to describe the characterizations of matrix operators. The first characterization is given in the following theorem.

THEOREM 3.4. *Let ϕ be an Orlicz function that satisfies the Δ_2-condition and $A_{nk}\in\mathcal{B}(L_\phi,Y)$ for each $n,k\in\mathbb{N}$. Then $A=(A_{nk})\in(W_{0,\phi},c(Y))$ if and only if*
(a) for each $u\in L_\phi$, there exists $A_k\in\mathcal{B}(L_\phi,Y)$ such that $A_{nk}u\rightarrow A_ku$ as $n\rightarrow\infty$, and
(b) $\sup_{n\in\mathbb{N},\,f\in U^*}\sum_{k=1}^{\infty}\|A_{nk}^*f\||a_k|<\infty$, for each $(a_k)\in w_{0,\phi}$.

Proof. Let $A\in(W_{0,\phi},c(Y))$ and take any $u\in L_\phi$. Then $e^ku\in W_{0,\phi}$ for any $k\in\mathbb{N}$. Hence, $A(e^ku)\in c(Y)$, i.e. $\lim_{n\to\infty}A_n(e^ku)=\lim_{n\to\infty}A_{nk}u$ exists. Since for each $n\in\mathbb{N}$, $A_{nk}\in\mathcal{B}(L_\phi,Y)$, then by Banach-Steinhaus's Theorem, the mapping $A_k: L_\phi\to Y$ where $A_k(u)=\lim_{n\to\infty}A_{nk}(u)$, belongs to $\mathcal{B}(L_\phi,Y)$. Thus we have (a).

For proving (b), let $(a_k)\in w_{0,\phi}$ and $n\in\mathbb{N}$. For each $k\in\mathbb{N}$ and $f\in U^*$, we have $A_{nk}^*f\in L_\phi^*$. Define the sequence $v=(v_k)$ as in (3) with A_{nk}^* in the place of T_k^*. By using (1), it is easy to show that $(v_k)\in W_{0,\phi}$, and hence $Av\in c(Y)$. Let M be a real number such that $\sup_n \|\sum_{k=1}^\infty A_{nk}v_k\|_Y \le M$. Since for any natural number s and $f \in U^*$, $\langle f, \sum_{k=1}^s A_{nk}v_k\rangle \le \|f\|\|\sum_{k=1}^s A_{nk}v_k\|_Y \le \|\sum_{k=1}^s A_{nk}v_k\|_Y$, then we get $\sum_{k=1}^\infty |a_k||A_{nk}^* f(u_k)| = \sum_{k=1}^\infty A_{nk}^* f(v_k) \le \|\sum_{k=1}^\infty A_{nk}v_k\|_Y \le M$, and this together with (2) yields (b).

For the sufficiency, let $(u_k) \in W_{0,\phi}$ and $f \in U^*$. By (b), $\sum_{k=1}^\infty \|A_{nk}^* f\||a_k| < \infty$, for each $(a_k) \in w_{0,\phi}$. Following Theorem 3.1, $\sum_{k=1}^\infty A_{nk}u_k$ is convergent for each n. Further, we will show that $Au \in c(Y)$.

Let $u \in L_\phi$ be arbitrary. Then, by (a) we have $|A_{nk}^* f(u) - A_k^* f(u)| = |\langle f, (A_{nk} - A_k)u\rangle| \le \|f\|\|(A_{nk} - A_k)u\|_Y \to 0$ as $n \to \infty$. For any $k \in \mathbb{N}$, let $f_k \in U^*$ such that $\|A_k u_k\|_Y = |\langle f_k, A_k u_k\rangle|$. First, we will prove that $\sum_{k=1}^\infty |A_{nk}^* f_k(u_k)|$ converges uniformly in n.

For each $k \in \mathbb{N}$, let $M_k = \sup_{n\in\mathbb{N}} |A_{nk}^* f_k(u_k)|$. Then there exists $n(k) \in \mathbb{N}$ such that $M_k \le |A_{n(k)k}^* f_k(u_k)| + \frac{1}{2^k}$. Since $(\|u_k\|_\phi)_{k\in\mathbb{N}} \in w_{0,\phi}$, then by (b) we have

$$\sum_{k=1}^\infty M_k \le \sum_{k=1}^\infty \left(|A_{n(k)k}^* f_k(u_k)| + \frac{1}{2^k}\right) \le \sum_{k=1}^\infty \left(\|A_{n(k)k}^* f_k\|\|u_k\|_\phi + \frac{1}{2^k}\right) < \infty.$$

Since $|A_{nk}^* f_k(u_k)| \le M_k$ for each k, then by the Weierstrass Test, $\sum_{k=1}^\infty |A_{nk}^* f_k(u_k)|$ converges uniformly in n. Furthermore,

$$\sum_{k=1}^\infty \|A_k u_k\|_Y = \sum_{k=1}^\infty |A_k^* f_k(u_k)| = \lim_{n\to\infty} \sum_{k=1}^\infty |A_{nk}^* f_k(u_k)| < \infty,$$

and hence, by completeness of Y, $\sum_{k=1}^\infty A_k u_k$ is convergent. Finally,

$$\left\|\sum_{k=1}^\infty (A_{nk} - A_k)u_k\right\|_Y \le \sum_{k=1}^\infty \|(A_{nk} - A_k)u_k\|_Y = \sum_{k=1}^\infty |(A_{nk}^* - A_k^*)f_k(u_k)| \to 0$$

whenever $n \to \infty$, i.e. $\lim_{n\to\infty} \sum_{k=1}^\infty A_{nk}u_k = \sum_{k=1}^\infty A_k u_k$. Thus, $(A_n u) \in c(Y)$ for each $(u_k) \in W_{0,\phi}$. Hence, $A = (A_{nk}) \in (W_{0,\phi}, c(Y))$. □

We also observe the following characterization.

Theorem 3.5. *Let ϕ be an Orlicz function that satisfies the Δ_2-condition and $A_{nk} \in \mathcal{B}(L_\phi, Y)$ for each $n, k \in \mathbb{N}$. Then $A = (A_{nk}) \in (W_\phi, c(Y))$ if and only if*
(a) for each k there exists $A_k \in \mathcal{B}(L_\phi, Y)$ such that $A_{nk}u \to A_k u$ as $n \to \infty$ for each $u \in L_\phi$,

(b) $\sup_{n\in\mathbb{N},\, f\in U^*} \sum_{k=1}^\infty \|A_{nk}^* f\||a_k| < \infty$, for each $(a_k) \in w_{0,\phi}$,

(c) $\sup_{n\in\mathbb{N},\, f\in U^*} \sum_{k=1}^\infty \|(A_{nk}^* - A_k^*)f\||a_k| < \infty$, for each $(a_k) \in w_{0,\phi}$.

Proof. Since $W_{0,\phi} \subset W_\phi$, then (a) and (b) are clear by Theorem 3.4. Further, we are going to prove (c).

Let $(a_k) \in w_{0,\phi}$, $f \in U^*$, and $s \in \mathbb{N}$. Define the sequence (v_k) as in (3) with T_k^* replaced by $A_{nk}^* - A_k^*$. It is clear that $(v_k) \in W_\phi$. So, it implies $\sup_{m,n\in\mathbb{N}} \|(A_n - A_m)v\|_Y \le M$ for some real number M. Since $\langle f, (A_{nk} - A_k)v_k\rangle = |a_k||(A_{nk}^* - A_k^*)fu_k|$ for every k, then

$$\sum_{k=1}^s |a_k||(A_{nk}^* - A_k^*)fu_k| = |\langle f, \sum_{k=1}^s (A_{nk}^* - A_k^*)v_k\rangle| \le \|\sum_{k=1}^s (A_{nk}^* - A_k^*)v_k\|_Y.$$

Further, by (a) we have

$$\left\|\sum_{k=1}^{s}(A_{nk}-A_k)\,v_k\right\|_Y=\lim_{m\to\infty}\left\|\sum_{k=1}^{s}(A_{nk}-A_{mk})\,v_k\right\|_Y.$$

So, $\sum_{k=1}^{\infty}|a_k||(A_{nk}^*-A_k^*)fu_k|\le M$ and (c) holds.

For sufficiency, let $(u_k)\in W_\phi$. Following (a), for every $f\in Y^*$ and $u\in L_\phi$ we have $|\langle f,(A_{nk}-A_k)u\rangle|\le\|f\|\|(A_{nk}-A_k)u\|_Y\to 0$ as $n\to\infty$, i.e. $\lim_{n\to\infty}A_{nk}^*f(u)=A_k^*f(u)$. Let us write

$$\sum_{k=1}^{\infty}A_{nk}u_k=\sum_{k=1}^{\infty}A_ku_k+\sum_{k=1}^{\infty}(A_{nk}-A_k)(u_k-u_0)+\sum_{k=1}^{\infty}(A_{nk}-A_k)u_0,$$

where $u_0\in L_\phi$ such that $(u_k-u_0)\in W_{0,\phi}$. We will examine that every term in the right-hand side is convergent.

For any k, let $f_k\in U^*$ such that $\|A_ku_k\|_Y=|A_k^*f_k(u_k)|$. It has been shown in the proof of Theorem 3.4, that the assumption (b) implies that $\sum_{k=1}^{\infty}|A_{nk}^*f_k(u_k)|$ converges uniformly in n. Hence, this result together with (a) yields

$$\begin{aligned}\sum_{k=1}^{\infty}\|A_ku_k\|_Y=\sum_{k=1}^{\infty}|A_k^*f_k(u_k)|&=\lim_{n\to\infty}\sum_{k=1}^{\infty}|A_{nk}^*f_k(u_k)|\\&\le\sup_{n\in\mathbb{N},\,f\in U^*}\sum_{k=1}^{\infty}\|A_{nk}^*f\|\|u_k\|_\phi<\infty.\end{aligned}$$

By the completeness of Y, it implies $\sum_{k=1}^{\infty}A_ku_k$ is convergent.

For any k, let $v_k=u_k-u_0$. By (c) and using the similar way as in the proof of Theorem 3.4, we can prove that $\sum_{k=1}^{\infty}|(A_{nk}^*-A_k^*)f_k(u_k-u_0)|$ is uniformly convergent in n. Hence,

$$\sum_{k=1}^{\infty}\|(A_{nk}-A_k)(u_k-u_0)\|_Y=\sum_{k=1}^{\infty}|(A_{nk}^*-A_k^*)f_k(u_k-u_0)|$$

converges to 0 as $n\to\infty$.

Finally, the sequence (u_k), where $u_k=u_0$ for every k, is in W_ϕ. Therefore, by (c), Theorem 3.1 and Theorem 3.2, $\sum_{k=1}^{\infty}(A_{nk}-A_k)u_0$ converges to 0 as $n\to\infty$. Hence, $\sum_{k=1}^{\infty}A_{nk}u_k$ converges as $n\to\infty$, i.e. the sequence $(\sum_{k=1}^{\infty}A_{nk}u_k)_{n\in\mathbb{N}}\in c(Y)$. ☐

For $1<p<\infty$, the function $\phi=|\cdot|^p$ is an Orlicz function and satisfies the Δ_2-condition. By using Lemma 2.2 and Theorem 3.5, we can prove the following corollary.

Corollary 3.6. *Let $1<p,q<\infty$ with $\frac{1}{p}+\frac{1}{q}=1$ and $A_{nk}\in\mathcal{B}(L_p,Y)$ for every $n,k\in\mathbb{N}$. Then $A=(A_{nk})\in(W_{|\cdot|^p},c(Y))$ if and only if*

(a) for each k there exists $A_k\in\mathcal{B}(L_q,Y)$ such that $A_{nk}u\to A_ku$ as $n\to\infty$ for all $u\in L_p$,

(b) $\sup_{n\in\mathbb{N},\,f\in U^*}\sum_{r=0}^{\infty}2^{r/p}\left(\sum_r\|A_{nk}^*f\|^q\right)^{1/q}<\infty$, and

(c) $\sup_{n\in\mathbb{N},\,f\in U^*}\sum_{r=0}^{\infty}2^{r/p}\left(\sum_r\|(A_{nk}^*-A_k^*)f\|^q\right)^{1/q}<\infty$.

Another main result is given in the following theorem.

Theorem 3.7. *Let ϕ be an Orlicz function that satisfies the Δ_2-condition and $A_{nk} \in \mathcal{B}(L_\phi, Y)$ for every $n, k \in \mathbb{N}$. The matrix $A = (A_{nk}) \in (\mathcal{W}_{\infty,\phi}, c(Y))$ if and only if $\sup_{n\in\mathbb{N},\, f\in U^*} \sum_{k=1}^{\infty} \|A^*_{nk} f\| |a_k| < \infty$ for each $(a_k) \in w_{\infty,\phi}$, and for any $k \in \mathbb{N}$, $u \in L_\phi$, there exists $A_k \in \mathcal{B}(L_\phi, Y)$ such that*
(a) $\lim_{n\to\infty} A_{nk} u = A_k u$,

(b) $\sup_{n\in\mathbb{N},\, f\in U^*} \sum_{k=1}^{\infty} \|(A^*_{nk} - A^*_k) f\| |a_k| < \infty$*, for each* $(a_k) \in w_{\infty,\phi}$,

(c) $\sup_{f\in U^*} \sum_{k=1}^{\infty} \|A^*_k f\| |a_k| < \infty$*, for each* $(a_k) \in w_{\infty,\phi}$.

Proof. For the necessity, let $A \in (W_{\infty,\phi}, c(Y))$ and $(a_k) \in W_{\infty,\phi}$. Construct the sequence $v = (v_k)$ as in (3) with T^*_k replaced by A^*_{nk}. It is clear that $(v_k) \in W_{\infty,\phi}$, hence $Av \in c(Y)$, and therefore there exists $M > 0$ such that $\sup_n \| \sum_{k=1}^{\infty} A_{nk} v_k \|_Y \leq M$. Since for each positive integer s and $f \in U^*$,

$$\sum_{k=1}^{s} |A^*_{nk} f(u_k)| |a_k| = \sum_{k=1}^{s} A^*_{nk} f v_k = \langle \sum_{k=1}^{s} A_{nk} v_k, f\rangle \leq \| \sum_{k=1}^{s} A_{nk} v_k \|_Y,$$

then by (2), we have $\sup_{n\in\mathbb{N},\, f\in U^*} \sum_{k=1}^{\infty} \|A^*_{nk} f\| |a_k| < \infty$. The condition (a) is clear by Theorem 3.5 and the fact that $W_\phi \subset W_{\infty,\phi}$. The proof of (b) is similar to the proof of Theorem 3.4 (b).

To prove (c), first we note that $\sum_{k=1}^{\infty} |A^*_{nk} f(v_k)|$ converges uniformly in n. It follows from $\sup_{n\in\mathbb{N},\, f\in U^*} \sum_{k=1}^{\infty} \|A^*_{nk} f\| |a_k| < \infty$. Hence, we have (c) from:

$$\sum_{k=1}^{\infty} |A^*_k f(u_k)| |a_k| = \sum_{k=1}^{\infty} |A^*_k f(v_k)| = \lim_{n\to\infty} \sum_{k=1}^{\infty} |A^*_{nk} f(v_k)| \leq \sup_n \|A_n v\|_Y < \infty.$$

For sufficiency, let $(u_k) \in W_{\infty,\phi}$. By Theorem 3.3 and following the hypotheses, $\sum_{k=1}^{\infty} A_{nk} u_k$ converges for each $n \in \mathbb{N}$. We will prove that $Au \in c(Y)$.

For any $k \in \mathbb{N}$, let $f_k \in U^*$ such that $\|(A_{nk} - A_k) u_k\|_Y = |(A^*_{nk} - A^*_k) f_k(u_k)|$. By (b), $\sum_{k=1}^{\infty} |(A^*_{nk} - A^*_k) f_k(u_k)|$ is uniformly convergent in n. Hence, by (a) we get $\sum_{k=1}^{\infty} |(A^*_{nk} - A^*_k) f_k(u_k)| \to 0$ as $n \to \infty$. Then $\sum_{k=1}^{\infty} \|(A_{nk} - A_k) u_k\|_Y = \sum_{k=1}^{\infty} |(A^*_{nk} - A^*_k) f_k(u_k)| \to 0$ as $n \to \infty$. Therefore, $\sum_{k=1}^{\infty} A_{nk} u_k \to \sum_{k=1}^{\infty} A_k u_k$ as $n \to \infty$. By Theorem 3.3 and (c), we get that $\sum_{k=1}^{\infty} A_k u_k$ converges, which means that $\lim_{n\to\infty} \sum_{k=1}^{\infty} A_{nk} u_k$ exists, i.e. $(\sum_{k=1}^{\infty} A_{nk} u_k)_{n\in\mathbb{N}} \in c(Y)$. □

As a straight consequence, we have the following corollary.

Corollary 3.8. *Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $A_{nk} \in \mathcal{B}(L_p, Y)$. Then $A = (A_{nk}) \in (W_{\infty,|\cdot|^p}, c(Y))$ if and only if $\sup_{n\in\mathbb{N},\, f\in U^*} \sum_{r=0}^{\infty} 2^{r/p} \left(\sum_r \|A^*_{nk} f\|^q \right)^{1/q} < \infty$, and for any $k \in \mathbb{N}$, $u \in L_p$, there exists $A_k \in \mathcal{B}(L_p, Y)$ such that*
(a) $\lim_{n\to\infty} A_{nk} u = A_k u$,

(b) $\sup_{n\in\mathbb{N},\, f\in U^*} \sum_{r=0}^{\infty} 2^{r/p} \left(\sum_r \|(A^*_{nk} - A^*_k) f\|^q \right)^{1/q} < \infty$*, and*

(c) $\sup_{f\in U^*} \sum_{r=0}^{\infty} 2^{r/p} \left(\sum_r \|A^*_k f\|^q \right)^{1/q} < \infty$.

REFERENCES

[1] F. Başar, B. Altay, E. Malkowsky, *Matrix transformations on some sequence spaces related to strong Cesáro summability and boundedness*, Publ. Math. Debrecen, **73(1–2)** (2008), 193—213.

[2] R. B. Darst, R. E. Zink, *A note on the definition of an Orlicz space*, Real Anal. Exch., **21(1)** (1995-96), 356–362.

[3] Haryadi, Supama, A. Zulijanto, *A generalization of Cesáro sequence spaces in the Orlicz space*, J. Phys., Conf. Ser., **1008** (2018), 012020.

[4] Haryadi, Supama, A. Zulijanto, *Kóthe-Toeplitz duals of the Cesàro sequence spaces defined on a generalized Orlicz space*, Glob. J. Pure Appl. Math., **14(4)** (2018), 591–601.

[5] M.A. Krasnosel'skii, Y.B. Rutickii, *Convex Functions and Orlicz Spaces*, Netherlands: P. Noordhoff Ltd., 1961.

[6] I.J. Maddox, *Generalized Köthe-Toeplitz Duals*, Int. J. Math. Math. Sci., **3** (1980), 423–432.

[7] I.J. Maddox, *Kuttner's theorem for operators*, Compos. Math., **29** (1974), 35–42.

[8] E. Malkowsky, V. Rakočević, *An introduction into the theory of sequence spaces and measures of noncompactness*, Zb. Rad. Beogr. **9** (2000), 143–234.

[9] E. Malkowsky, V.I. Veličković, *Topologies of some new sequence spaces, their duals, and the graphical representations of neighborhoods*, Topology Appl., **158** (2011), 1369–1380.

[10] E. Malkowsky, F. Özger, V. I. Veličković, *Some spaces related to Cesàro sequence spaces and an application to crystallography*, MATCH Commun. Math. Comput Chem., **70(3)** (2013),867–884.

[11] E. Malkowsky, V. I. Veličkovič, *Some new sequence spaces, their duals and a connection with Wulff's crystal*, MATCH Commun. Math. Comput. Chem., **67(3)** (2012), 589-–607.

[12] M. M. Rao, Z. D. Ren, *Theory of Orlicz Spaces*. New York: Marcel Dekker Inc., 1991.

[13] Y. Yilmaz, M.K. Ozdemir, *Köthe-Toeplitz duals of some vector-valued Orlicz sequence space*, Soochow J. Math., **31(3)** (2005), 389–402.

(received 05.01.2021; in revised form 07.03.2022; available online 21.01.2023)

Department of Computer Science Universitas Muhammadiyah Palangkaraya, Indonesia

E-mail: haryadi_ump@yahoo.co.id

Department of Mathematics Universitas Gadjah Mada, Yogyakarta 55281, Indonesia

E-mail: supama@ugm.ac.id

Department of Mathematics Universitas Gadjah Mada, Yogyakarta 55281, Indonesia

E-mail: atokzulijanto@ugm.ac.id