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GENERALIZED *V-Ric* VECTOR FIELDS ON CONTACT PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we study contact pseudo-Riemannian manifold M admitting generalized V-*Ric* vector field. Firstly, for pseudo-Riemannian manifold, it is proved that V is an infinitesimal harmonic transformation if M admits V-*Ric* vector field. Secondly, we prove that an η-Einstein K-contact pseudo-Riemannian manifold admitting a generalized V-*Ric* vector field is either Einstein or has scalar curvature $r = \frac{2n\varepsilon(2n-1)}{4n-1}$. Finally, we consider a contact pseudo-Riemannian (κ, μ)-manifold with a generalized V-*Ric* vector field.

1. Introduction

A vector field V on a pseudo-Riemannian manifold (M, g) is said to be concircular [5] if it satisfies

$$\nabla_X V = \nu X, \tag{1}$$

where ν denotes a smooth function on M. If ν in (1) is non-constant, then we say V is non-trivial concircular. A concircular vector field V is called a concurrent vector field [13] if the function ν in (1) is equal to one.

A vector field V on a pseudo-Riemannian manifold (M, g) is said to be conformal if $\pounds_V g = 2\nu g$, where $\pounds$ denotes a Lie derivative. Particularly, we call V homothetic and Killing if ν is constant and zero, respectively. The authors in [14, 15] studied the geometry of conformal and Killing vector fields on contact Riemannian manifolds.

A generalized V-*Ric* vector field was introduced by Hinterleitner and Kiosak [9] and it is defined by

$$\nabla_X V = \nu QX, \quad \text{for any } X \text{ on } M, \tag{2}$$

where Q is the Ricci operator. Einstein manifolds are characterized by the proportionality of Ricci tensor Ric to the metric tensor. So, for Einstein manifold, the condition of vector field V being concircular could equally be defined by (2). We say that V is

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V-*Ric* vector field when ν in (2) is constant. If ν is non-constant, then we say that the vector field V is proper generalized V-*Ric* vector field. Moreover, when $\nu = 0$, the vector field V is covariantly constant (also Killing). If we take $V = 0$, then (2) is meaningless and hence, we always assume that that generalized Ricci vector filed V is non-zero. In [10], it is shown that V-*Ric* vector fields are closely related to the Ricci flow introduced by Hamilton [8]. Vashpanov et al. [17] studied geodesic mapping of spaces with V-*Ric* vector fields and obtained a solution for integrability conditions of these equations. Recently, Wang and Wu [19] studied generalized V-*Ric* vector fields on K-contact Riemannian manifolds.

An almost contact pseudo-Riemannian manifolds are a natural generalization of almost contact Riemannian manifolds (also called almost contact metric structure). The study of contact structure endowed with pseudo-Riemannian metric were first considered by Takahashi [16], who focused on Sasakian case. Calvaruso and Perrone [2] undertook a systematic study of contact structures with pseudo-Riemannian associated metrics. Such manifolds have been enormously studied under various points of view (see [2, 11, 12, 18] and references cited therein). In this paper, we study the generalized V-*Ric* vector fields within the framework of contact pseudo-Riemannian manifolds.

2. Preliminaries

A $(2n+1)$-dimensional differentiable manifold M endowed with a $(1,1)$-tensor field φ, a vector field ξ (called Reeb vector field) and a 1-form η, is called an almost contact manifold if these tensors satisfy the following relations

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0. \tag{3}$$

It follows from (3) that the rank of φ is $2n$. We refer to [1] for more information.

If an almost contact manifold is equipped with a pseudo-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \tag{4}$$

where $\varepsilon = \pm 1$, and therefore $g(\xi, \xi) = \varepsilon$ (the Reeb vector field cannot be light-like), then $(M, \varphi, \eta, \xi, g)$ is called *almost contact pseudo-Riemannian manifold* or *almost contact pseudo-metric manifold*. The signature of associated metric g is either $(2m + 1, 2n - m)$ or $(2m, 2n - 2m - 1)$, according to whether the Reeb vector field ξ is space-like or time-like. From the relation (4), it can be seen that $\eta(X) = \varepsilon g(\xi, X)$, $g(\varphi X, Y) = -g(X, \varphi Y)$. An almost contact pseudo-Riemannian manifold is called a *contact pseudo-Riemannian manifold* if $d\eta = \Phi$, where $\Phi(X, Y) = g(X, \varphi Y)$ is a fundamental 2-form.

We define a self-adjoint $(1,1)$-tensor field h and ℓ by

$$hX = \frac{1}{2}(\pounds_\xi \varphi)X, \quad \text{and} \quad \ell X = R(X, \xi)\xi, \tag{5}$$

where $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is the curvature tensor. The sign convention of R is opposite to the one used in [3, 11]. The operators in (5) satisfy the following equalities

$$h\xi = 0 = \ell\xi, \quad h\varphi = -\varphi h, \quad tr(h) = tr(\varphi h) = 0. \tag{6}$$

We now accumulate some formulas which are valid for a contact pseudo-Riemannian manifold [2, 12];

$$\nabla_X \xi = -\varepsilon\varphi X - \varphi h X, \tag{7}$$

$$(\nabla_\xi h)X = \varphi X - h^2\varphi X + \varphi R(\xi, X)\xi, \tag{8}$$

$$\mathrm{tr}\,(\nabla\varphi) = 2n\xi, \quad \mathrm{div}\,\xi = \mathrm{div}\,\eta = 0,$$

where tr is the trace operator and *div* is the divergence operator.

If Reeb vector field ξ of contact pseudo-Riemannian manifold M is Killing (equivalently $h = 0$), then M is called *K-contact pseudo-Riemannian manifold.* A *Sasakian pseudo-Riemannian manifold* is a contact pseudo-Riemannian manifold whose almost contact structure (φ, ξ, η) is normal, i.e., the almost complex structure J on $M \times \mathbb{R}$ defined by $J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right)$, is integrable, where f is a real-valued function and t is the coordinate on $\mathbb{R}$. Moreover, a contact pseudo-Riemannian manifold M is Sasakian if and only if $(\nabla_X\varphi)Y = g(X, Y)\xi - \varepsilon\eta(Y)X$. Any Sasakian pseudo-Riemannian manifold is always K-contact and the converse also holds when $n = 1$, i.e., for 3-dimensional spaces. It is worthwhile to mention that, on a Sasakian pseudo-Riemannian manifold we obtain

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X. \tag{9}$$

In contact Riemannian case, the above equation shows that the manifold is Sasakian, but this is not valid in the case of contact pseudo-Riemannian [11]. However, the following lemma holds.

Lemma 2.1 ([11]). *A K-contact pseudo-Riemannian manifold M is Sasakian if and only if the curvature tensor R satisfies* (9).

3. Generalized V-*Ric* vector field on contact pseudo-Riemannian manifolds

In this section we study generalized V-*Ric* vector field on contact pseudo-Riemannian manifolds. First, we prove the following result.

Theorem 3.1. *Let M be a pseudo-Riemannian manifold. If M admits a V-Ric vector field, then V is an infinitesimal harmonic transformation.*

Proof. To prove this result, we follow the technique of Ghosh [7]. From (2), it can be easily obtained that

$$(\pounds_V g)(X, Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V) = 2\nu\mathrm{Ric}\,(X, Y). \tag{10}$$

Differentiating the above equation covariantly along Z and using (7), we get

$$(\nabla_Z \pounds_V g)(X, Y) = 2\nu(\nabla_Z \mathrm{Ric}\,)(X, Y). \tag{11}$$

According to Yano's book [20], the following commutation formula holds
$(\pounds_V \nabla_Z g - \nabla_Z \pounds_V g - \nabla_{[V,Z]} g)(X,Y) = -g((\pounds_V \nabla)(Z,X),Y) - g((\pounds_V \nabla)(Z,Y),X)$.
The parallelism of the pseudo-Riemannian metric transforms the above equation to $(\nabla_Z \pounds_V g)(X,Y) = g((\pounds_V \nabla)(Z,X),Y) + g((\pounds_V \nabla)(Z,Y),X)$. By virtue of (11), it follows from aforesaid equation that

$$g((\pounds_V \nabla)(Z,X),Y) + g((\pounds_V \nabla)(Z,Y),X) = 2\nu(\nabla_Z \mathrm{Ric}\,)(X,Y). \qquad (12)$$

Cyclic rotation of X, Y and Z in (12) and simple calculation yield

$$g((\pounds_V \nabla)(X,Y),Z) = \nu\{(\nabla_X \mathrm{Ric}\,)(Y,Z) + (\nabla_Y \mathrm{Ric}\,)(Z,X) - (\nabla_Z \mathrm{Ric}\,)(X,Y)\}.$$

Setting $X = Y = E_i$ (where $\{E_i\}_{i=1}^n$ is a local pseudo-orthonormal basis) in the last equation and summing over i, we find

$$\sum_{i=1}^{n} \varepsilon_i (\pounds_V \nabla)(E_i, E_i) = 0, \qquad (13)$$

where $\varepsilon_i = g(E_i, E_i)$ and we have employed $\operatorname{div} Q = \frac{1}{2} Dr$. According to Duggal and Sharma [4], $(\pounds_V \nabla)(X,Y) = \nabla_X \nabla_Y V - \nabla_{\nabla_X Y} V + R(V,X)Y$. From the previous equation, it follows that

$$\sum_{i=1}^{n} \varepsilon_i (\pounds_V \nabla)(E_i, E_i) = \sum_{i=1}^{n} \varepsilon_i (\nabla_{E_i} \nabla_{E_i} V - \nabla_{\nabla_{E_i} E_i} V) + \sum_{i=1}^{n} \varepsilon_i R(V, E_i) E_i. \qquad (14)$$

From the equations (13) and (14), we easily obtain $0 = QV - \tilde{\Delta} V$, where $\tilde{\Delta} V = \sum_i \varepsilon_i (\nabla_{\nabla_{E_i} E_i} V - \nabla_{E_i} \nabla_{E_i} V)$ is the so-called rough Laplacian of V. In this seting, it is rightful to reveal that a vector field V is an infinitesimal harmonic transformation if and only if $QV = \tilde{\Delta} V$ (see [3]). □

A pseudo-Riemannian manifold (M, g) is said to admit a Yamabe soliton if there exist a vector field V and a constant λ such that

$$(\pounds_V g)(X,Y) = 2(r - \lambda) g(X,Y). \qquad (15)$$

The Yamabe soliton was introduced in [8] as the selfsimilar solution of the Yamabe flow. A Yamabe soliton is said to be shrinking, steady or expanding according to $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively.

Theorem 3.2. *If a contact pseudo-Riemannian manifold M admits a Yamabe soliton with soliton vector field V being V-Ric vector field, then it is Einstein.*

Proof. Assume that the soliton vector field V is V-*Ric* vector field, i.e., $\nabla_X V = \nu Q X$. Therefore, it can be easily obtained from (10) and (15) that $\mathrm{Ric} = \frac{r-\lambda}{\nu} g$. □

In what follows we consider some special contact pseudo-Riemannian manifolds admitting generalized V-*Ric* vector field.

Lemma 3.3. *If a K-contact pseudo-Riemannian manifold M admits a generalized V-Ric vector field, then the following relation holds*

$$V - 2nD\nu = 0. \qquad (16)$$

Proof. It was obtained in [11, Theorem 3.1] that in a K-contact pseudo-Riemannian manifold, the Reeb vector field ξ is an eigenvector of the Ricci operator, i.e., $Q\xi = 2n\varepsilon\xi$. Take covariant derivative of this equation along X and make use of (7) to get

$$(\nabla_X Q)\xi = \varepsilon Q\varphi X - 2n\varphi X \tag{17}$$

We know that ξ is Killing on K-contact pseudo-Riemannian manifold, so $\pounds_\xi \mathrm{Ric} = 0$. It follows that $0 = (\pounds_\xi Q)X = \pounds_\xi(QX) - Q(\pounds_\xi X) = (\nabla_\xi Q)X - \nabla_{QX}\xi + Q(\nabla_X \xi)$. By virtue of (7), we obtain from the previous equation that $(\nabla_\xi Q)X = \varepsilon(Q\varphi X - \varphi QX)$.

We have assumed that V is a generalized V-*Ric* vector field. Covariant derivative of (2) implies that $\nabla_Y \nabla_X V = Y(\nu)QX + \nu(\nabla_Y Q)X$. It directly follows that

$$R(X,Y)V = X(\nu)QY - Y(\nu)QX + \nu\{(\nabla_X Q)Y - (\nabla_Y Q)X\}. \tag{18}$$

Since ξ Killing vector field, then by (7) we have

$$R(X,\xi)Y = \varepsilon\nabla_X\varphi Y - \varepsilon\varphi\nabla_X Y = \varepsilon(\nabla_X\varphi)Y. \tag{19}$$

Replacing Y in (18) by ξ and utilizating (17) and (19), we obtain

$$\begin{aligned} &-\varepsilon g((\nabla_X\varphi)Y, V) \\ &\quad= 2nX(\nu)\eta(Y) - \xi(\nu)\mathrm{Ric}\,(X,Y) + \nu\{\varepsilon g(\varphi QX, Y) - 2ng(\varphi X, Y)\}, \end{aligned} \tag{20}$$

where we have employed $Q\xi = 2n\varepsilon\xi$. Replacing X by φX and Y by φY in (20), adding the resulting equation with (20) and then call up the well-known formula (see [2, Lemma 4.3]) $(\nabla_X\varphi)Y + (\nabla_{\varphi X}\varphi)\varphi Y = 2g(X,Y)\xi - \eta(Y)\{\varepsilon X + \varepsilon\eta(X)\xi\}$, we obtain

$$\begin{aligned} &-\varepsilon\{2g(X,Y)g(\xi,V) - \eta(Y)(\varepsilon g(X,V) + \varepsilon\eta(X)g(\xi,V))\} \\ &\quad= 2nX(\nu)\eta(Y) - \xi(\nu)\mathrm{Ric}\,(X,Y) - \xi(\nu)\mathrm{Ric}\,(\varphi X, \varphi Y) \\ &\qquad+ \nu\{\varepsilon g(\varphi QX + Q\varphi X, Y) + 4ng(X, \varphi Y)\}. \end{aligned}$$

Anti-symmetrizing the preceding equation, we achieve

$$\begin{aligned} &\eta(Y)\{g(X,V) + \varepsilon\eta(V)\eta(X)\} - \eta(X)\{g(Y,V) + \varepsilon\eta(V)\eta(Y)\} \\ &\quad= 2n\{X(\nu)\eta(Y) - Y(\nu)\eta(X)\} + \nu\{2\varepsilon g(\varphi QX + Q\varphi X, Y) - 8ng(\varphi X, Y)\} \end{aligned} \tag{21}$$

Now, replacing Y in (21) by ξ and using (4) provides

$$V - \eta(V)\xi - 2n(D\nu - \varepsilon\xi(\nu)\xi) = 0. \tag{22}$$

Taking the derivative of (22) along X and utilizating (2), $h = 0$, (7), we obtain

$$\nu QX + g(V,\varphi X)\xi - 2n\varepsilon\eta(X)\xi + \varepsilon\eta(V)\varphi X - 2n(\nabla_X D\nu - \varepsilon X(\xi(\nu))\xi + \xi(\nu)\varphi X) = 0,$$

where we have used $Q\xi = 2n\varepsilon\xi$. Applying the Poincare lemma (i.e., $d^2 = 0$), remembering that $g(\nabla_X D\nu, Y)$ is symmetric, we get

$$\begin{aligned} 0 = \varepsilon g(V,\varphi X)\eta(Y) &- \varepsilon g(V,\varphi Y)\eta(X) + 2\varepsilon\eta(V)g(\varphi X, Y) \\ &+ 2n(X(\xi(\nu))\eta(Y) - Y(\xi(\nu))\eta(X) + 2\xi(\nu)g(X,\varphi Y)). \end{aligned}$$

Replacing X by φX and Y by φY in the aforesaid equation, we see that $(2n\xi(\nu) - \varepsilon\eta(V))d\eta(X,Y) = 0$. Since $d\eta$ is non-vanishing everywhere on M, the last equation shows that $\eta(V) = 2n\varepsilon\xi(\nu)$, which, when inserted in (22), implies (16). □

THEOREM 3.4. *If an η-Einstein K-contact pseudo-Riemannian manifold M of dimension > 3 admits a generalized V-Ric vector field, then either M is Einstein and*

V is concircular (also conformal), or the scalar curvature is $r = \frac{2n\varepsilon(2n-1)}{4n-1}$.

Proof. On η-Einstein K-contact pseudo-Riemannian manifold M of dimension > 3, we have the expression of Ricci operator as

$$QX = \left(\frac{r}{2n} - \varepsilon\right) X + \varepsilon \left(2n + 1 - \frac{r\varepsilon}{2n}\right) \eta(X)\xi, \tag{23}$$

where r is the constant scalar curvature (see [12]). In view of constancy of r, contracting X in (18) and using the formula $\operatorname{div} Q = \frac{1}{2}Dr$ we obtain $QV = QD\nu - rD\nu$. As a result of (23), it follows that

$$\left(\frac{r}{2n} - \varepsilon\right) V + \varepsilon \left(2n + 1 - \frac{r\varepsilon}{2n}\right) \eta(V)\xi = \left(\frac{r}{2n} - \varepsilon - r\right) D\nu + \left(2n + 1 - \frac{r\varepsilon}{2n}\right) \xi(\nu)\xi.$$

In view of (16), the afore mentioned equation reduces to

$$\left(\frac{r(4n-1)}{2n} - (2n-1)\varepsilon\right) D\nu = (2n-1) \left(2n + 1 - \frac{r\varepsilon}{2n}\right) \xi(\nu)\xi. \tag{24}$$

Differentiating (24) covariantly along X, making use of (7) provides

$$\left(\frac{r(4n-1)}{2n} - (2n-1)\varepsilon\right) \nabla_X D\nu = (2n-1) \left(2n + 1 - \frac{r\varepsilon}{2n}\right) (X(\xi(\nu))\xi - \xi(\nu)\varepsilon\varphi X).$$

Since $g(\nabla_X D\nu, Y) = g(\nabla_Y D\nu, X)$, it follows from above equation that

$$\left(2n + 1 - \frac{r\varepsilon}{2n}\right) \varepsilon\{X(\xi(\nu))\eta(Y) - Y(\xi(\nu))\eta(X) - 2\xi(\nu)g(\varphi X, Y)\} = 0. \tag{25}$$

In view of (25), we have either $r = 2n\varepsilon(2n+1)$ or $r \neq 2n\varepsilon(2n+1)$. First, we consider $r = 2n\varepsilon(2n+1)$ and in this case the manifold is Einstein, i.e., $QX = 2n\varepsilon X$. This, inserted in (2), shows that V is concircular (also conformal). In the later case, we have from (25) that $X(\xi(\nu))\eta(Y) - Y(\xi(\nu))\eta(X) - 2\xi(\nu)g(\varphi X, Y) = 0$. Taking X and Y orthogonal to ξ in the foregoing equation yields $\xi(\nu) = 0$, as $d\eta \neq 0$ on M. This together with (24) entails that either $r = \frac{2n\varepsilon(2n-1)}{4n-1}$ or ν is constant. If we assume that ν is constant, then from the relation (16) we get a contradiction as V is zero. So that the only choice is $r = \frac{2n\varepsilon(2n-1)}{4n-1}$. □

4. Generalized V-Ric vector field on (κ, μ)-contact pseudo-Riemannian manifolds

In [6], Ghaffarzadeh and Faghfouri introduced the notion of contact pseudo-Riemannian (κ, μ)-manifold. According to them a contact pseudo-Riemannian (κ, μ)-manifold is a contact pseudo-Riemannian manifold whose curvature tensor R satisfies

$$R(X, Y)\xi = \varepsilon\kappa\{\eta(Y)X - \eta(X)Y\} + \varepsilon\mu\{\eta(Y)hX - \eta(X)hY\}, \tag{26}$$

for some real numbers κ, μ. For contact pseudo-Riemannian (κ, μ)-manifold we have the following relations (see [6]):

$$h^2 = (\varepsilon\kappa - 1)\varphi^2, \tag{27}$$

$$Q\xi = 2n\kappa\xi. \tag{28}$$

Lemma 4.1. [6] *In any contact pseudo-Riemannian* (κ, μ)*-manifold* M *of dimension* $2n+1$*, the Ricci operator* Q *of* M *can be expressed as*

$$QX = \varepsilon(2(n-1) - n\mu)X + (2(n-1)+\mu)hX \\ +(2(1-n)\varepsilon + 2n\kappa + n\mu\varepsilon)\eta(X)\xi, \tag{29}$$

where $\varepsilon\kappa < 1$*. Further, the scalar curvature of* M *is* $2n(2(n-1)\varepsilon - n\mu\varepsilon + \kappa)$*.*

We are now prepared for the following outcome.

Theorem 4.2. *If contact pseudo-Riemannian* (κ, μ)*-manifold* M *with* $\varepsilon\kappa < 1$ *admits a generalized* V*-Ric vector field, then one of the following cases holds.*
(i) V *is a parallel vector field.*
(ii) The curvature tensor satisfies $R(X,Y)\xi = 0$*.*
(iii) A smooth function ν *satisfies* $\nu = \frac{\varepsilon}{2n\kappa}\left(1 - \frac{r}{2n\kappa}\right)\xi(\xi(\nu))$*.*

Proof. Differentiation of (28) along X and utilization of (7) yields

$$(\nabla_X Q)\xi = Q(\varepsilon\varphi + \varphi h)X - 2n\kappa(\varepsilon\varphi + \varphi h)X. \tag{30}$$

Taking the scalar product of (18) with ξ and employing (28), (30) provides

$$g(R(X,Y)V,\xi) = 2n\varepsilon\kappa\{X(\nu)\eta(Y) - Y(\nu)\eta(X)\} + \nu\{g(Q(\varepsilon\varphi + \varphi h)X, Y) \\ - g(Q(\varepsilon\varphi + \varphi h)Y, X) - 4n\varepsilon\kappa g(\varphi X, Y)\}. \tag{31}$$

Replacing Y in (31) by ξ and utilizating (3), (26) and (28) implies

$$\varepsilon\kappa(\eta(V)\xi - V) - \varepsilon\mu hV = 2n\kappa(\varepsilon D\nu - \xi(\nu)\xi). \tag{32}$$

In view of constancy of r, contracting X in (18) and calling up the formula $\operatorname{div} Q = \frac{1}{2}Dr$ gives that $QV = QD\nu - rD\nu$. This together with (29) shows that

$$\varepsilon(2(n-1) - n\mu)V + (2(n-1)+\mu)hV + (2(1-n)\varepsilon + 2n\kappa + n\mu\varepsilon)\eta(V)\xi \\ = (\varepsilon(2(n-1) - n\mu) - r)D\nu + (2(n-1)+\mu)hD\nu + \varepsilon(2(1-n)\varepsilon + 2n\kappa + n\mu\varepsilon)\xi(\nu)\xi.$$

Taking the scalar product of the aforementioned equation with ξ and taking the first term of (6) gives

$$\eta(V) = \varepsilon\left(1 - \frac{r}{2n\kappa}\right)\xi(\nu). \tag{33}$$

Inserting (33) in (32), it follows that

$$\left((2n+1)\kappa - \frac{r}{2n}\right)\xi(\nu)\xi - \varepsilon\kappa V - \varepsilon\mu hV - 2n\varepsilon\kappa D\nu = 0. \tag{34}$$

Replacing X by φX and Y by φY in the foregoing equation and using $R(\varphi X, \varphi Y)\xi = 0$ (follows from (26)) and (3), we obtain $\nu\{\varepsilon(Q\varphi + \varphi Q)X - \varphi QhX - hQ\varphi X - 4n\varepsilon\varphi X\} = 0$. By virtue of (27) and (29), it can be obtained from the above equation that $\nu\{\varepsilon\kappa(\mu-2) - \mu(n+1)\} = 0$. Thus, from the above relation, we have that either $\nu = 0$ or $\varepsilon\kappa(\mu-2) - \mu(n+1) = 0$. If we consider $\nu = 0$, then from (2) we conclude that V parallel vector field. Next, we consider

$$\varepsilon\kappa(\mu-2) - \mu(n+1) = 0. \tag{35}$$

Differentiating (34) along X and taking (2), (7) provides

$$\varepsilon\left((2n+1)\kappa - \frac{r}{2n}\right)\{X(\xi(\nu))\eta(Y) - \xi(\nu)(\varepsilon g(\varphi X, Y) + g(\varphi hX, Y))\}$$

$$- \varepsilon\kappa\nu g(QX, Y) - \varepsilon\mu g((\nabla_X h)Y, V) - \varepsilon\mu g(hQX, Y) - 2n\varepsilon\kappa g(\nabla_X D\nu, Y) = 0.$$

Putting $X = Y = \xi$ in the foregoing equation and utilizating (6), (8), (28) implies

$$\kappa \left(\varepsilon \left(1 - \frac{r}{2n\kappa}\right) \xi(\xi(\nu)) - 2n\kappa\nu \right) = 0. \tag{36}$$

If we suppose that $\kappa = 0$, it follows from (35) that $\mu = 0$. Hence $R(X, Y)\xi = 0$ for any vector X, Y on M. Suppose $\kappa \neq 0$; then from (36) one can conclude that a smooth function ν satisfy $\nu = \frac{\varepsilon}{2n\kappa}\left(1 - \frac{r}{2n\kappa}\right)\xi(\xi(\nu))$. □

In the Riemannian setting, if a contact manifold M satisfies $R(X, Y)\xi = 0$ then it is locally flat in dimension 3 and in higher dimensions it is locally isometric to the trivial bundle $E^{n+1} \times S^n(4)$ (see [1]). Thus, we have the following corollary.

Corollary 4.3. *If a contact Riemannian (κ, μ)-manifold M with $\kappa < 1$ admits a generalized V-Ric vector field, then one of the following cases holds.*
(i) V is a Killing vector field.

(ii) M is locally flat in dimension 3 and in higher dimensions it is locally isometric to the trivial bundle $E^{n+1} \times S^n(4)$.

(iii) A smooth function ν satisfies $\nu = \frac{1}{2n\kappa}\left(1 - \frac{r}{2n\kappa}\right)\xi(\xi(\nu))$.

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