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APPROXIMATION SPACES VIA IDEALS AND GRILLS

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Abstract. In this paper, we use the notions of lower set $L_R(A)$ and the upper set $U_R(A)$ to define the interior operator int_R^A and the closure operator cl_R^A associated with a set A in an approximation space (X, R). These operators generate an approximation topological space different from the generated Nano topological space in (X, R). Ideal approximation spaces (X, R, ℓ) based on an ideal ℓ joined to the approximation space (X, R) are introduced as well. The approximation continuity and the ideal approximation continuity are defined. The lower separation axioms $T_i, i = 0, 1, 2$ are introduced in the approximation spaces and also in the ideal approximation spaces. Examples are given to explain the definitions. Connectedness in approximation spaces and ideal connectedness are introduced and the differences between them are explained. The interior and the closure operators are deduced using a grill $\mathcal{G}$ defined on (X, R), yielding the same results.

1. Introduction

Rough sets were defined by Pawlak in an approximation space [5] as an extension of set theory and refer to the uncertainty of intelligent systems characterized by insufficient and incomplete information. Basically, rough sets are defined as a function of an equivalence relation R defined on a universal finite set X, which is considered as the main concept for the lower and upper approximation of a subset $A \subseteq X$. The pair (X, R) was called an approximation space based on an equivalence relation on X. Many kinds of generalizations of Pawlak's rough set were obtained by replacing the equivalence relation with an arbitrary binary relation. On the other hand, relations between rough sets and topological spaces were studied by many authors [6, 7]. It was proved that the lower and upper approximation operators derived by a reflexive and transitive relation were exactly the interior and closure operators in a topology. Many research works were introduced for the ordinary case with rough sets with some medical applications as in [2, 3].

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L. Thivagar et. al. introduced in [8] a nano topological space with respect to a subset A of X defined by lower and upper approximations of A. Recently, many researchers have used topological approaches in the study of rough sets and their applications.

In this paper, we introduce a new generalization of rough sets based on an equivalence relation R over the notion of a coset of an element x in the approximation space (X, R). The definition of an ideal ℓ on the approximation space led to new definitions of rough sets in the ideal approximation spaces (X, R, ℓ). We introduce the interior and closure operators based on the ideal ℓ and the lattice $\mathcal{G}$ (the dual concept), but neither generate a topological space and have no relation to the associated nano topological space. Approximation continuity and ideal approximation continuity are introduced, and examples are given to show the differences. Ideal open and ideal closed sets are also given. The lower separation axioms $T_i, i = 0, 1, 2$ are introduced in the approximation spaces and also in the ideal approximation spaces. Examples are given to explain the definitions. The notion of approximation connectedness and ideal approximation connectedness are defined. The implications are proved by examples.

In the course of the paper, let X be a finite set of objects as a universal set, let 2^X denote all subsets of X, and let R be an equivalence relation on X. Then let the pair (X, R) be an approximation space. The coset $[x]$ of an element $x \in X$ is given by $[x](y) = R(x, y)$, $\forall y \in X$. For a set $K \subseteq X$, the lower $(L_R(K))$, the upper $(U_R(K))$, and the boundary region $B_R(K)$ are approximation sets defined as follows (see [8]):

$$L_R(K) = \bigcup_{x \in X} \{[x] : [x] \subseteq K\},$$

$$U_R(K) = \bigcup_{x \in X} \{[x] : [x] \cap K \neq \phi\}, \quad B_R(K) = U_R(K) - L_R(K).$$

$L_R(A), U_R(A)$ and $B_R(A)$ are then called lower, upper and boundary region approximation sets associated with the set A in 2^X and based on the equivalence relation R in an approximation space (X, R).

Lemma 1.1 ([8]). *For any sets $A, B \in 2^X$ we get that:*

(i) $L_R(A) \subseteq A \subseteq U_R(A)$,

(ii) $L_R(\phi) = U_R(\phi) = \phi$ *and* $L_R(X) = U_R(X) = X$,

(iii) $L_R(A \cap B) = L_R(A) \cap L_R(B)$ *and* $U_R(A \cup B) = U_R(A) \cup U_R(B)$,

(iv) $A \subseteq B$ *implies that* $L_R(A) \subseteq L_R(B)$ *and* $U_R(A) \subseteq U_R(B)$,

(v) $U_R(A \cap B) \subseteq U_R(A) \cap U_R(B)$ *and* $L_R(A \cup B) \supseteq L_R(A) \cup L_R(B)$,

(vi) $(U_R(A))^c = L_R(A^c)$ *and* $(L_R(A))^c = U_R(A^c)$,

(vii) $U_R(L_R(A)) \supseteq L_R(L_R(A)) = L_R(A)$,

(viii) $L_R(U_R(A)) \subseteq U_R(U_R(A)) = U_R(A)$.

Definition 1.2. Associated with $A \subseteq X$ in an approximation space (X, R), the interior operator $\text{int}_R^A : 2^X \to 2^X$ is given as follows:

$$\text{int}_R^A(B) = L_R(A) \cap L_R(B) \quad \forall\, B \neq X \ \text{ and } \ \text{int}_R^A(X) = X.$$

Also, the closure operator $\mathrm{cl}_R^A : 2^X \to 2^X$ is given as follows:

$$\mathrm{cl}_R^A(B) = (L_R(A))^c \cup U_R(B) \quad \forall\, B \neq \phi \ \text{ and } \ \mathrm{cl}_R^A(\phi) = \phi.$$

The proof of the following lemma is easy with the help of Lemma 1.1.

LEMMA 1.3. *(i)* $\mathrm{int}_R^A(\phi) = \mathrm{cl}_R^A(\phi) = \phi$, $\mathrm{int}_R^A(X) = \mathrm{cl}_R^A(X) = X$,

(ii) $\mathrm{int}_R^A(B) \subseteq B \subseteq \mathrm{cl}_R^A(B) \forall B \in 2^X$,

(iii) $B \subseteq C \Longrightarrow \mathrm{int}_R^A(B) \subseteq \mathrm{int}_R^A(C)$, $\mathrm{cl}_R^A(B) \subseteq \mathrm{cl}_R^A(C) \forall B, C \in 2^X$,

(iv) $\mathrm{int}_R^A(B \cap C) = \mathrm{int}_R^A(B) \cap \mathrm{int}_R^A(C), \mathrm{int}_R^A(B \cup C) \supseteq \mathrm{int}_R^A(B) \cup \mathrm{int}_R^A(C) \quad \forall B, C \in 2^X$,

(v) $\mathrm{cl}_R^A(B \cap C) \subseteq \mathrm{cl}_R^A(B) \cap \mathrm{cl}_R^A(C), \mathrm{cl}_R^A(B \cup C) = \mathrm{cl}_R^A(B) \cup \mathrm{cl}_R^A(C) \quad \forall B, C \in 2^X$,

(vi) $\mathrm{int}_R^A(\mathrm{int}_R^A(B)) = \mathrm{int}_R^A(B) \quad \forall B \in 2^X$,

(vii) $\mathrm{cl}_R^A(\mathrm{cl}_R^A(B)) = \mathrm{cl}_R^A(B) \quad \forall B \in 2^X$.

Hence, these operators generate a topology in the approximation space defined by

$$T_R^A = \{B \in 2^X : B = \mathrm{int}_R^A(B)\} \text{ or} \tag{1}$$

$$T_R^A = \{B \in 2^X : B^c = \mathrm{cl}_R^A(B^c)\}.$$

Note that $$\mathrm{cl}_R^A(U_R(B)) = \mathrm{cl}_R^A(B), \quad \mathrm{int}_R^A(L_R(B)) = \mathrm{int}_R^A(B), \forall B \in 2^X, \tag{2}$$

and moreover, $\mathrm{int}_R^A(B^c) = (\mathrm{cl}_R^A(B))^c$, $\mathrm{cl}_R^A(B^c) = (\mathrm{int}_R^A(B))^c$, $\forall B \in 2^X$.

A nano topology τ_A was defined in (X, R) as $\tau_A = \{\phi, X, L_R(A), U_R(A), B_R(A)\}$ (see [8]). There is no relation between the nano topology τ_A constructed on X and the approximation topology T_R^A generated by the interior operator as in (1). In Example 2.8, we get a nano open set but not in T_R^A, and in Example 2.13 we get a set in T_R^A but not nano open set.

2. Ideal approximation spaces

A non-empty collection ℓ of subsets of a non-empty set X is said to be an ideal [4] on X if it satisfies the following conditions:
(i) If $A \subseteq B$ and $B \in \ell$, then $A \in \ell$ for all $A, B \in 2^X$.

(ii) If $A \in \ell$ and $B \in \ell$, then $(A \cup B) \in \ell$ for all $A, B \in 2^X$.
In order to exclude the trivial case where the ideal coincides with the set of all subsets of X, it is generally assumed that $X \notin \ell$. In this case, ℓ is called a proper ideal on X.

The triple (X, R, ℓ) is called an ideal approximation space. Since ℓ is a non-empty collection, then the coarsest ideal is $\ell = \{\phi\}$.

DEFINITION 2.1. Let (X, R, ℓ) be an ideal approximation space and $A \in 2^X$. Then, the local set $\Phi_A(B)(R, \ell)$ of a set $B \in 2^X$ with respect to A is defined by:

$$\Phi_A(B)(R, \ell) = \bigcap \{G \in 2^X : (B - G) \in \ell,\ \mathrm{cl}_R^A(G) = G\}.$$

For short, we will write $\Phi_A(B)$ or $\Phi_A(B)(\ell)$ instead of $\Phi_A(B)(R, \ell)$.

Corollary 2.2. *Let (X, R, ℓ°) be an ideal approximation space, $A \in 2^X$ where ℓ° is the trivial ideal $\{\phi\}$ on X. Then, for each $B \in 2^X$, we have $\Phi_A(B) = \mathrm{cl}_R^A(B)$.*

Proof. Since $B \subseteq \mathrm{cl}_R^A(B), \mathrm{cl}_R^A(\mathrm{cl}_R^A(B)) = \mathrm{cl}_R^A(B)$, then $\Phi_A(B) \subseteq \Phi_A(\mathrm{cl}_R^A(B)) = \mathrm{cl}_R^A(B)$ from the definition of the operator Φ_A.

If $\ell^\circ = \{\phi\}$, then $\Phi_A(B) = \bigcap\{K \in 2^X : B - K = \phi, \mathrm{cl}_R^A(K) = K\}$, that is, $\Phi_A(B) = \bigcap\{K \in 2^X : B \subseteq K, \mathrm{cl}_R^A(K) = K\}$.

Now, we show that in case of $\ell = \{\phi\}$, we have $\Phi_A(B) = \mathrm{cl}_R^A(K)$, and thus we need to prove that $\Phi_A(B) \supseteq \mathrm{cl}_R^A(B)$. If we supposed that $\mathrm{cl}_R^A(B) \not\subseteq K = \Phi_A(B)$, then $B \subseteq K, \mathrm{cl}_R^A(K) = K$ so that $\mathrm{cl}_R^A(B) \not\subseteq K$. But $B \subseteq K, \mathrm{cl}_R^A(K) = K$ implies that $\mathrm{cl}_R^A(B) \subseteq \mathrm{cl}_R^A(K) = K$, which is a contradiction. Hence, $\Phi_A(B) = \mathrm{cl}_R^A(B)$. □

For any $H \in \ell$ and ℓ an ideal on X in the approximation space (X, R), we get, from the definition of the operator $\Phi_A(G)$ for any $G \subseteq X$, that $\Phi_A(H) = \phi$.

Let (X, R, ℓ) be an ideal approximation space associated with $A \in 2^X$ and ℓ an ideal on X. Then, for each $G, H \in 2^X$: Since $B \subseteq \mathrm{cl}_R^A(B), \mathrm{cl}_R^A(\mathrm{cl}_R^A(B)) = \mathrm{cl}_R^A(B)$, then $\Phi_A(B) \subseteq \Phi_A(\mathrm{cl}_R^A(B)) = \mathrm{cl}_R^A(B)$ from the definition of the operator Φ_A.

If $\ell^\circ = \{\phi\}$, then $\Phi_A(B) = \bigcap\{K \in 2^X : B - K = \phi, \mathrm{cl}_R^A(K) = K\}$, that is, $\Phi_A(B) = \bigcap\{K \in 2^X : B \subseteq K, \mathrm{cl}_R^A(K) = K\}$.
Now, we show that in case of $\ell = \{\phi\}$, we have $\Phi_A(B) = \mathrm{cl}_R^A(K)$, and thus we need to prove that $\Phi_A(B) \supseteq \mathrm{cl}_R^A(B)$. If we supposed that $\mathrm{cl}_R^A(B) \not\subseteq K = \Phi_A(B)$, then $B \subseteq K, \mathrm{cl}_R^A(K) = K$ so that $\mathrm{cl}_R^A(B) \not\subseteq K$. But $B \subseteq K, \mathrm{cl}_R^A(K) = K$ implies that $\mathrm{cl}_R^A(B) \subseteq \mathrm{cl}_R^A(K) = K$, which is a contradiction. Hence, $\Phi_A(B) = \mathrm{cl}_R^A(B)$.

Proposition 2.3. *(i) $G \subseteq H$ implies $\Phi_A(G) \subseteq \Phi_A(H)$.*

(ii) If $\ell \subseteq \ell^*$ and ℓ^* is an ideal on X, then $\Phi_A(G)(\ell) \supseteq \Phi_A(G)(\ell^*)$.

(iii) $\mathrm{int}_R^A(\Phi_A(G)) \subseteq \Phi_A(G) = \mathrm{cl}_R^A(\Phi_A(G)) \subseteq \mathrm{cl}_R^A(G)$.

(iv) $\Phi_A(G) = \Phi_A(\Phi_A(G))$.

(v) $\Phi_A(G) \cup \Phi_A(H) = \Phi_A(G \cup H)$.

(vi) $\Phi_A(G) \cap \Phi_A(H) \supseteq \Phi_A(G \cap H)$.

Proof. (i) Suppose that $\Phi_A(G) \not\subseteq \Phi_A(H)$; then there exists $W \in 2^X$ with $H - W \in \ell$ and $\mathrm{cl}_R^A(W) = W$ such that $\Phi_A(G) \not\subseteq W$. Since $G \subseteq H$, then $G - W \subseteq H - W$ and then $G - W \in \ell$, $\mathrm{cl}_R^A(W) = W$. Thus, $\Phi_A(G) \subseteq W$, which is a contradiction and hence $\Phi_A(G) \subseteq \Phi_A(H)$.

(ii) Let $H \in 2^X$ be such that $G - H \in \ell^*$, $\mathrm{cl}_R^A(H) = H$. Since $\ell \subseteq \ell^*$, then $G - H \in \ell$, $\mathrm{cl}_R^A(H) = H$. Hence, from the definition of the operator $\Phi_A(K)$ for any $K \in 2^X$, we get that $\Phi_A(G)(\ell) \supseteq \Phi_A(G)(\ell^*)$.

(iii) $\mathrm{int}_R^A(\Phi_A(G)) \subseteq \Phi_A(G) = \mathrm{cl}_R^A(\Phi_A(G))$, directly. Since $\Phi_A(G) \subseteq \mathrm{cl}_R^A(G)$, then $\Phi_A(G) = \mathrm{cl}_R^A(\Phi_A(G)) \subseteq \mathrm{cl}_R^A(G)$.

(iv) $\Phi_A(\Phi_A(G)) \subseteq \mathrm{cl}_R^A(\Phi_A(G))$ is clear.

Let $\Phi_A(\Phi_A(G)) = K$, that is, $\Phi_A(G) - K \in \ell$, $\mathrm{cl}_R^A(K) = K$. Suppose that $\Phi_A(G) = H \not\subseteq K = \Phi_A(\Phi_A(G))$. Then, $G - H \in \ell$, $\mathrm{cl}_R^A(H) = H$, which means that

$H - K \in \ell$ and $G - H \in \ell$, and then $G - K \subseteq (G - H) \cup (H - K) \in \ell, \mathrm{cl}_R^A(K) = K$, and thus $\Phi_A(G) = H \subseteq K$, which is a contradiction. So, $\Phi_A(G) \subseteq \Phi_A(\Phi_A(G))$.

From (iii), we have $\Phi_A(\Phi_A(G)) = \mathrm{cl}_R^A(\Phi_A(\Phi_A(G))) \subseteq \mathrm{cl}_R^A(\Phi_A(G)) = \Phi_A(G)$. Hence, $\Phi_A(\Phi_A(G)) = \mathrm{cl}_R^A(\Phi_A(\Phi_A(G))) \subseteq \mathrm{cl}_R^A(\Phi_A(G)) = \Phi_A(G)$. Thus, $\Phi_A(G) = \Phi_A(\Phi_A(G))$.

(v) From (i), $\Phi_A(G) \subseteq \Phi_A(G \cup H)$, $\Phi_A(H) \subseteq \Phi_A(G \cup H)$. Hence, $\Phi_A(G) \cup \Phi_A(H) \subseteq \Phi_A(G \cup H)$.

Now, suppose that $\Phi_A(G) \cup \Phi_A(H) \not\supseteq \Phi_A(G \cup H)$, then there exist $B, W \subseteq X$ such that $G - B \in \ell$, $H - W \in \ell$, $\mathrm{cl}_R^A(B) = B$, $\mathrm{cl}_R^A(W) = W$ with $B \cup W \not\supseteq \Phi_A(G \cup H)$. But $G \cup H - (B \cup W) \in \ell$, and $\mathrm{cl}_R^A(B \cup W) = B \cup W$. Hence, $\Phi_A(G \cup H) \supseteq B \cup W$, which is a contradiction, and thus $\Phi_A(G) \cup \Phi_A(H) = \Phi_A(G \cup H)$.

(vi) Obvious. □

Definition 2.4. Let (X, R, ℓ) be an ideal approximation space and $A \in 2^X$. Then, for any $G \subseteq X$, define the operators $\mathrm{cl}_\Phi^A, \mathrm{int}_\Phi^A : 2^X \to 2^X$ as follows:

$$(\mathrm{cl}_\Phi^A)(G) = \mathrm{cl}_R^A(G) \cup \Phi_A(U_R(A)), \quad (\mathrm{int}_\Phi^A)(G) = \mathrm{int}_R^A(G) \cap (\Phi_A(U_R(A)))^c, \quad \forall G \in 2^X.$$

cl_Φ^A and int_Φ^A are operators from 2^X into 2^X associated with a specific set A and an ideal ℓ in the approximation space (X, R).

Now, if $\ell = \ell^\circ$, then from (2), Corollary 2.2 and Lemma 1.3, $(\mathrm{cl}_\Phi^A)(G) = \mathrm{cl}_R^A(G \cup A) \supseteq \mathrm{cl}_R^A(G) = \Phi_A(G) = \mathrm{cl}_R^A(\Phi_A(G))$ and $(\mathrm{int}_\Phi^A)(G) = \mathrm{int}_R^A(G \cap A^c) \subseteq \mathrm{int}_R^A(G) = (\Phi_A(G^c))^c = \mathrm{int}_R^A((\Phi_A(G^c))^c) \forall G \in 2^X$.

Proposition 2.5. *Let (X, R, ℓ) be an ideal approximation space with $A \in 2^X$ fixed.*

(i) $(\mathrm{int}_\Phi^A)(G) \subseteq \mathrm{int}_R^A(G) \subseteq G \subseteq \mathrm{cl}_R^A(G) \subseteq (\mathrm{cl}_\Phi^A)(G)$.

(ii) $\mathrm{cl}_\Phi^A(G^c) = ((\mathrm{int}_\Phi^A)(G))^c$ *and* $\mathrm{int}_\Phi^A(G^c) = ((\mathrm{cl}_\Phi^A)(G))^c$.

(iii) $\mathrm{int}_\Phi^A(G \cup H) \supseteq \mathrm{int}_\Phi^A(G) \cup \mathrm{int}_\Phi^A(H)$, $\mathrm{cl}_\Phi^A(G \cap H) \subseteq \mathrm{cl}_\Phi^A(G) \cap \mathrm{cl}_\Phi^A(H)$.

(iv) $\mathrm{int}_\Phi^A(G \cap H) = \mathrm{int}_\Phi^A(G) \cap \mathrm{int}_\Phi^A(H)$, $\mathrm{cl}_\Phi^A(G \cup H) = \mathrm{cl}_\Phi^A(G) \cup \mathrm{cl}_\Phi^A(H)$.

(v) $\mathrm{cl}_\Phi^A(\mathrm{cl}_\Phi^A(G)) \supseteq (\mathrm{cl}_\Phi^A)(G)$ *and* $\mathrm{int}_\Phi^A(\mathrm{int}_\Phi^A(G)) \subseteq (\mathrm{int}_\Phi^A)(G)$.

(vi) If $G \subseteq H$, then $\triangle(G) \subseteq \triangle(H)$ for all $\triangle \in \{\mathrm{cl}_\Phi^A, \mathrm{int}_\Phi^A\}$.

These ideal approximation operators cl_Φ^A, int_Φ^A do not generate a topology on X.

In order to define separation axioms in the approximation spaces, we found that for any $x \notin A$, $G \in 2^X$ cannot be found such that $x \in \mathrm{int}_R^A(G)$ because $\mathrm{int}_R^A(G) = L_R(G) \cap L_R(A) \subseteq A$. Thus, in defining the separation axioms, we will rely only on the elements of A.

Definition 2.6. Let (X, R, ℓ) be an ideal approximation space and $A \in 2^X$. Then:

(i) An ideal approximation space (X, R, ℓ) (resp. an approximation space (X, R)) is called an ideal-T_0 (resp. T_0) if for every $x \neq y \in A$, there exists $G \subseteq X$ with $x \in \mathrm{int}_\Phi^A(G)$ (resp. $x \in \mathrm{int}_R^A(G)$) such that $y \notin G$ or there exists $H \subseteq X$ with $y \in \mathrm{int}_\Phi^A(H)$ (resp. $y \in \mathrm{int}_R^A(H)$) such that $x \notin H$.

(ii) An ideal approximation space (X, R, ℓ) (resp. an approximation space (X, R)) is called an ideal-T_1 (resp. T_1) if for every $x \neq y \in A$, there exist $G, H \subseteq X$ with $x \in \mathrm{int}_{\Phi}^{A}(G)$, $y \in \mathrm{int}_{\Phi}^{A}(H)$ (resp. $x \in \mathrm{int}_{R}^{A}(G)$, $y \in \mathrm{int}_{R}^{A}(H)$) such that $y \notin G$ and $x \notin H$.

(iii) An ideal approximation space (X, R, ℓ) (resp. an approximation space (X, R)) is called an ideal-T_2 (resp. T_2) if for every $x \neq y \in A$, there exist $G, H \subseteq X$ with $x \in \mathrm{int}_{\Phi}^{A}(G)$, $y \in \mathrm{int}_{\Phi}^{A}(H)$ (resp. $x \in \mathrm{int}_{R}^{A}(G)$, $y \in \mathrm{int}_{R}^{A}(H)$) such that $G \cap H = \phi$.

Remark 2.7. It is clear from $\mathrm{int}_{\Phi}^{A} \subseteq \mathrm{int}_{R}^{A}$ that:

$$\begin{tikzcd}
\text{ideal-}T_2 \arrow[r] \arrow[d] & \text{ideal-}T_1 \arrow[r] \arrow[d] & \text{ideal-}T_0 \arrow[d] \\
T_2 \arrow[r] & T_1 \arrow[r] & T_0
\end{tikzcd}$$

Example 2.8. Let $X = \{a, b, c\}$, $X|R = \{\{a\}, \{b, c\}\}$, $A = \{a, b\}$. Then, $L_R(A) = \{a\}, U_R(A) = X, (L_R(A))^c = \{b, c\}$. That is, $L_R(\{a\}) = L_R(\{a, b\}) = L_R(\{a, c\}) = \{a\}$, $L_R(\{b, c\}) = \{b, c\}$ and $L_R(\{b\}) = L_R(\{c\}) = \phi$. Hence, for $a \neq b \in A$, we see that (X, R) is T_0 but not T_1 or T_2 approximation space.

Now, the subsets satisfying $\mathrm{cl}_{R}^{A}(K) \equiv U_R(K) \cup (L_R(A))^c = K$ are only $\phi, X, \{b, c\}$. If we suppose the ideal ℓ is given by $\ell = \{\phi, \{b\}\}$, then $\Phi_A(U_R(A)) = \Phi_A(X) = X$. Hence, $(\mathrm{int}_{\Phi}^{A})(K) = \mathrm{int}_{R}^{A}(K) \cap (\phi_A(U_R(A)))^c = \phi$ for any subset $K \subseteq X$. Thus, (X, R, ℓ) is not ideal-T_i, for all $i = 0, 1, 2$.

If we suppose the ideal ℓ is given by $\ell = \{\phi, \{a\}\}$, then $\Phi_A(U_R(A)) = \Phi_A(X) = \{b, c\}$. Hence, $(\mathrm{int}_{\Phi}^{A})(K) = \mathrm{int}_{R}^{A}(K) \cap (\phi_A(U_R(A)))^c = \{a\}$ for any subset $K \in \{\{a\}, \{a, b\}, \{a, c\}\}$. That means for any $a \neq b \in A$, we have $(\mathrm{int}_{\Phi}^{A})(K) = \mathrm{int}_{R}^{A}(K) = \{a\}$, and moreover (X, R, ℓ) is an ideal-T_0 but not ideal-T_1 or ideal-T_2.

Here, the nano topology associated with A is given by $\tau_A = \{\phi, X, \{a\}, \{b, c\}\}$, and thus for any $a \neq b \in A$, we get disjoint nano open sets, and moreover (X, τ_A) is a nano T_i-space; $i = 0, 1, 2$. Note that $\{b, c\}$ is a nano open set but $\mathrm{int}_{R}^{A}(\{b, c\}) = \phi \neq \{b, c\}$.

Recall that a mapping $f : (X, R) \to (Y, R^*)$ is said to be approximation continuous (App-cont.) if $\mathrm{int}_{R}^{A}(f^{-1}(H)) \supseteq f^{-1}(\mathrm{int}_{R^*}^{B}(H))$, $\forall H \subseteq Y$. It is equivalent to $\mathrm{cl}_{R}^{A}(f^{-1}(H)) \subseteq f^{-1}(\mathrm{cl}_{R^*}^{B}(H))$, $\forall H \subseteq Y$. Now with respect to $A \subseteq X$ and $B \subseteq Y$, let us call a mapping $f : (X, R) \to (Y, R^*, \ell)$ ideal approximation continuous (ideal App-cont.) provided that $\mathrm{int}_{\Phi}^{A}(f^{-1}(H)) \supseteq f^{-1}(\mathrm{int}_{R^*}^{B}(H))$, $\forall H \subseteq Y$. It is easily shown that it is equivalent to $\mathrm{cl}_{\Phi}^{A}(f^{-1}(H)) \subseteq f^{-1}(\mathrm{cl}_{R^*}^{B}(H))$, $\forall H \subseteq Y$. Also, let us call $f : (X, R) \to (Y, R^*)$ approximation open (App-open) provided that $\mathrm{int}_{R^*}^{B}(f(G)) \supseteq f(\mathrm{int}_{R}^{A}(G))$, $\forall G \subseteq X$,

$f : (X, R, \ell) \to (Y, R^*)$ is ideal approximation open (ideal App-open) provided that $\mathrm{int}_{\Phi}^{B}(f(G)) \supseteq f(\mathrm{int}_{R}^{A}(G))$, $\forall G \subseteq X$.

Clearly, every ideal approximation continuous (resp. ideal approximation open) mapping is approximation continuous (resp. approximation open) as well (from Proposition 2.5 (i)) but the converse is not true.

Theorem 2.9. *Let $(X, R, \ell), (Y, R^*, \ell^*)$ be two ideal approximation spaces associated with $A \subseteq X$, $B \subseteq Y$, respectively and $f : (X, R, \ell) \to (Y, R^*, \ell^*)$ be an injective ideal approximation continuous mapping with $f(A) = B$. Then, X is an ideal T_i-space if Y is an ideal T_i-space, $i = 0, 1, 2$.*

Proof. Since $x \neq y$ in A implies that $f(x) \neq f(y)$ in B, and if Y is an ideal T_2-space, then there exist $C = f(G), D = f(K) \in 2^Y$ for some $G, K \subseteq X$ with $f(x) \in \mathrm{int}^B_\Phi(C)$, $f(y) \in \mathrm{int}^B_\Phi(D)$ such that $C \cap D = \phi$, that is, $x \in f^{-1}(\mathrm{int}^B_\Phi(C))$, $y \in f^{-1}(\mathrm{int}^B_\Phi(D))$, and then $x \in f^{-1}(\mathrm{int}^B_{R^*}(f(G)))$, $y \in f^{-1}(\mathrm{int}^B_{R^*}(f(K)))$. Since f is ideal App-cont., then $x \in \mathrm{int}^A_\Phi(f^{-1}(f(G)))$, $y \in \mathrm{int}^A_\Phi(f^{-1}(f(K)))$.

That is, there exist $G, K \in 2^X$ with $x \in \mathrm{int}^A_\Phi(G)$, $y \in \mathrm{int}^A_\Phi(K)$ and $G \cap K = f^{-1}(C \cap D) = \phi$. Hence, (X, R, ℓ) is an ideal T_2-space. Other cases are similar. □

Theorem 2.10. *Let $(X, R, \ell), (Y, R^*, \ell^*)$ be two ideal approximation spaces associated with $A \subseteq X$, $B \subseteq Y$, respectively and $f : (X, R, \ell) \to (Y, R^*, \ell^*)$ be a surjective ideal approximation open mapping with $f^{-1}(B) = A$. Then, Y is an ideal T_i-space if X is an ideal T_i-space, $i = 0, 1, 2$.*

Proof. Since $p \neq q$ in B implies that there are $x \in f^{-1}(p)$ and $y \in f^{-1}(q)$, $f(x) = p, f(y) = q$ with $x \neq y$ in A, and if X is an ideal T_2-space, then there exist $G = f^{-1}(C), K = f^{-1}(D) \in 2^X$ for some $C, D \subseteq Y$ with $x \in \mathrm{int}^A_\Phi(G)$, $y \in \mathrm{int}^A_\Phi(K)$ such that $G \cap K = \phi$, that is, $p \in f(\mathrm{int}^A_\Phi(G))$, $q \in f(\mathrm{int}^A_\Phi(K))$, and then $p \in f(\mathrm{int}^A_R(f^{-1}(C)))$, $q \in f(\mathrm{int}^A_R(f^{-1}(D)))$. Since f is ideal App-open, then $p \in \mathrm{int}^B_\Phi(C)$, $q \in \mathrm{int}^B_\Phi(D)$.

That is, there exist $C = f(G)$, $D = f(K)$ with $p \in \mathrm{int}^B_\Phi(C)$, $q \in \mathrm{int}^B_\Phi(D)$ and $f^{-1}(C \cap D) = \phi$, f is surjective and thus $C \cap D = \phi$. Hence, (Y, R^*, ℓ^*) is an ideal T_2-space. The other cases are similar. □

Definition 2.11. Let (X, R) be an approximation space and $A \subseteq X$. Then:

(i) Two sets $B, C \in 2^X$ are called ideal approximation separated (resp. approximation separated) if $\mathrm{cl}^A_\Phi(B) \cap C = B \cap \mathrm{cl}^A_\Phi(C) = \phi$ (resp. $\mathrm{cl}^A_R(B) \cap C = B \cap \mathrm{cl}^A_R(C) = \phi$).

(ii) A set $G \in 2^X$ is called ideal approximation disconnected (resp. approximation disconnected) set if there exist ideal approximation separated (resp. approximation separated) sets $B, C \in 2^X$ such that $B \cup C = G$. A set G is called ideal approximation connected (ideal App-conn.) (resp. approximation connected (App-conn.)) if it is not ideal approximation disconnected (ideal App-disconn.) (resp. approximation disconnected (App-disconn.)).

(iii) (X, R, ℓ) is called an ideal approximation disconnected space if there exist ideal approximation separated sets $B, C \in 2^X$, such that $B \cup C = X$. An ideal approximation space(X, R, ℓ) is called ideal approximation connected if it is not ideal approximation disconnected.

(iv) (X, R) is called an approximation disconnected space if there exist approximation separated sets $B, C \in 2^X$ such that $B \cup C = X$. An approximation space(X, R) is called approximation connected if it is not approximation disconnected.

Remark 2.12. Any two ideal approximation separated sets B, C in 2^X are approximation separated as well (from that $\mathrm{cl}_R^A(W) \subseteq \mathrm{cl}_\Phi^A(W) \forall W \in 2^X$). That is, ideal approximation disconnectedness implies approximation disconnectedness and thus, approximation connectedness implies ideal approximation connectedness.

The following is an example proving that not every approximation disconnected set is an ideal approximation disconnected set.

Example 2.13. Let $X = \{a, b, c, d, e\}$, $X|R = \{\{a, b\}, \{c\}, \{d, e\}\}$ and $A = \{a, b, d, e\}$. Then, $L_R(A) = U_R(A) = \{a, b, d, e\}$ and $(L_R(A))^c = \{c\}$.

Let $B = \{a\}$, $C = \{e\}$, then $U_R(B) = \{a, b\}$, $U_R(C) = \{d, e\}$, and then $\mathrm{cl}_R^A(B) = U_R(B) \cup (L_R(A))^c = \{a, b, c\}$ and $\mathrm{cl}_R^A(C) = U_R(C) \cup (L_R(A))^c = \{c, d, e\}$. Hence, $\mathrm{cl}_R^A(B) \cap C = \mathrm{cl}_R^A(C) \cap B = \phi$, and thus B, C are two approximation separated sets in X. That is, the subset $K = \{a, e\}$ is approximation disconnected.

Define an ideal ℓ on X as follows: $\ell = \{\phi, \{a\}, \{c\}, \{a, c\}\}$. Then, $\Phi_A(U_R(A)) = \bigcap\{H \subseteq X : U_R(A) - H \in \ell,\ \mathrm{cl}_R^A(H) = H\}$, and there are only these subsets $\{c\}$, $\{a, b, c\}$, $\{c, d, e\}$, ϕ, X satisfying $\mathrm{cl}_R^A(H) = H$ for any set H equal to some of these subsets. But from the definition of the ideal, we get that only $H = X$ satisfies the condition $U_R(A) - X \in \ell$, and thus $\Phi_A(U_R(A)) = X$. Hence, $\mathrm{cl}_\Phi^A(B) = \mathrm{cl}_\Phi^A(C) = X$, $\mathrm{cl}_\Phi^A(B) \cap C \neq \phi$. Also, $\mathrm{cl}_\Phi^A(C) \cap B \neq \phi$. Thus, the subset $K = \{a, e\}$ is not ideal separated with the sets B, C, and is not ideal separated at all because we have found that $\mathrm{cl}_\Phi^A(B) = \mathrm{cl}_\Phi^A(C) = X$.

The nano topology τ_A is given by $\tau_A = \{\phi, X, A\}$, and hence the nano closed sets are only $\{c\}$, ϕ, X, which means it is impossible to find two separated sets in (X, τ_A), which means that (X, τ_A) is not nano disconnected space.

Note that a set $K = \{a, b\}$ satisfying $K = \mathrm{int}_R^A(K)$ but K is not nano open set.

Proposition 2.14. *Let (X, R, ℓ) be an ideal approximation space associated with $A \subseteq X$ and let $B \in 2^X$. Then, the following are equivalent.*
(i) B is ideal approximation connected set.

(ii) If C, D are ideal approximation separated sets with $B \subseteq (C \cup D)$, then $B \cap C = \phi$ or $B \cap D = \phi$.

(iii) If C, D are ideal approximation separated sets with $B \subseteq (C \cup D)$, then $B \subseteq C$ or $B \subseteq D$.

Remark 2.15. If ℓ and ℓ^* are independent ideals on X and Y respectively, then the mapping $f : (X, R, \ell) \to (Y, R^*, \ell^*)$ is still not ideal approximation continuous in general, even if we take f to be a bijective map with respect to $A \subseteq X$ and $f(A) \in 2^Y$ and the relations R on X and R^* on Y where $R^* = R \circ (f^{-1} \times f^{-1}) = (f \times f)(R)$. This special case itself could be as an example of an approximation continuous mapping but which is not ideal approximation continuous in general.

Theorem 2.16. *Let (X, R, ℓ), (Y, R^*, ℓ^*) associated with $A \subseteq X$ and $B \in 2^Y$, respectively be ideal approximation spaces and $f : (X, R, \ell) \to (Y, R^*, \ell^*)$ is an ideal approximation continuous mapping. Then, $f(G) \in 2^Y$ is an ideal approximation connected set if G is an ideal approximation connected set in X.*

Proof. Let $C, D \in 2^Y$ be ideal approximation separated sets with $f(G) = C \cup D$. That is, $\mathrm{cl}_\Phi^B(C) \cap D = \mathrm{cl}_\Phi^B(D) \cap C = \phi$, and thus $\mathrm{cl}_{R^*}^B(C) \cap D = \mathrm{cl}_{R^*}^B(D) \cap C = \phi$. Then, $G \subseteq (f^{-1}(C) \cup f^{-1}(D))$, and from f is ideal approximation continuous, we get $\mathrm{cl}_\Phi^A(f^{-1}(C)) \cap f^{-1}(D) \subseteq f^{-1}(\mathrm{cl}_{R^*}^B(C)) \cap f^{-1}(D) = f^{-1}(\mathrm{cl}_{R^*}^B(C) \cap D) = f^{-1}(\phi) = \phi$, and similarly, we have $\mathrm{cl}_\Phi^A(f^{-1}(D)) \cap f^{-1}(C) \subseteq f^{-1}(\mathrm{cl}_{R^*}^B(D)) \cap f^{-1}(C) = f^{-1}(\mathrm{cl}_{R^*}^B(D) \cap C) = f^{-1}(\phi) = \phi$.

Hence, $f^{-1}(C)$ and $f^{-1}(D)$ are ideal approximation separated sets in X so that $G \subseteq (f^{-1}(C) \cup f^{-1}(D))$. But from Proposition 2.14 (iii), we get that $G \subseteq f^{-1}(C)$ or $G \subseteq f^{-1}(D)$, which means that $f(G) \subseteq C$ or $f(G) \subseteq D$. Thus, since G is an ideal approximation connected set in X, and again from Proposition 2.14 (iii), we get that $f(G)$ is ideal approximation connected in Y. □

The implications in the following diagram are satisfied whenever f is ideal approximation continuous (ideal App-cont.).

$$\begin{tikzcd}
G \text{ is App-conn.} \arrow[r] \arrow[d] \arrow[dr] & G \text{ is ideal App-conn.} \arrow[d] \arrow[dl] \\
f(G) \text{ is App-conn.} \arrow[r] & f(G) \text{ is ideal App-conn.}
\end{tikzcd}$$

Just the implications in the following diagram are satisfied whenever f is approximation continuous (App-cont.).

$$\begin{tikzcd}
G \text{ is App-conn.} \arrow[r] \arrow[d] \arrow[dr] & G \text{ is ideal App-conn.} \\
f(G) \text{ is App-conn.} \arrow[r] & f(G) \text{ is ideal App-conn.}
\end{tikzcd}$$

3. Grill approximation spaces

The idea of Grills on a topological space was firstly introduced by Choquet [1]. The concept of grills has proved to be a powerful supporting and useful mathematical tool like nets and filters for a deeper insight into further studying of some topological notions such as proximity spaces, closure spaces and compactness.

A collection $\mathcal{G}$ of 2^X is called a grill [1] on X if $\mathcal{G}$ satisfies the following conditions:

(i) $A \in \mathcal{G}$ and $A \subseteq B$ implies that $B \in \mathcal{G}$,

(ii) $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Remark 3.1. Let X be a non-empty set and $\mathcal{G} \subseteq 2^X$. Then, $\mathcal{G}$ is a grill on X iff $\ell(\mathcal{G}) = \{B \in 2^X : B \notin \mathcal{G}\}$ is an ideal on X.

Conversely, let X be a non-empty set and $\ell \subseteq 2^X$. Then, ℓ is an ideal on X iff $\mathcal{G}(\ell) = \{B^c \in 2^X : B \in \ell\}$ is a grill on X.

Definition 3.2. Let (X, R) be an approximation space associated with $A \in 2^X$ and $\mathcal{G}$ a grill on X. Define a mapping $\Psi_A : 2^X \to 2^X$ as follows $\Psi_A(B) = \bigcap\{G \in 2^X : B - G \notin \mathcal{G}, \mathrm{cl}_R^A(G) = G\}$ for all $B \in 2^X$. Then, the mapping Ψ_A is called the operator associated with the grill $\mathcal{G}$ in the approximation space (X, R) with respect to $A \in 2^X$.

If $\mathcal{G} = 2^X - \{\phi\}$ we have that $\Psi_A(G) = \mathrm{cl}_R^A(G)$. The triple $(X, R, \mathcal{G})$ is called a grill approximation space.

For a grill approximation space $(X, R, \mathcal{G})$ associated with $A \in 2^X$, define the mapping $\mathrm{cl}_\Psi^A : 2^X \to 2^X$ as follows $\mathrm{cl}_\Psi^A(G) = \mathrm{cl}_R^A(G) \cup \Psi_A(U_R(A))$, $\forall G \in 2^X$.

Also, the mapping $\mathrm{int}_\Psi^A : 2^X \to 2^X$ is defined by $\mathrm{int}_\Psi^A(G) = \mathrm{int}_R^A(G) \cap (\Psi_A(U_R(A)))^c$, $\forall G \in 2^X$. Now, if $\mathcal{G} = 2^X - \{\phi\}$, then

$$(\mathrm{cl}_\Psi^A)(G) = \mathrm{cl}_R^A(G \cup A) \supseteq \mathrm{cl}_R^A(G) = \Psi_A(G) = \mathrm{cl}_R^A(\Psi_A(G)), \quad \text{and}$$
$$(\mathrm{int}_\Psi^A)(G) = \mathrm{int}_R^A(G \cap A^c) \subseteq \mathrm{int}_R^A(G) = (\Psi_A(G^c))^c = \mathrm{int}_R^A((\Psi_A(G^c))^c) \quad \forall G \in 2^X.$$

Definition 3.3. Let $(X, R, \mathcal{G})$ be a grill approximation space associated with $A \in 2^X$. Then,

(i) $B \subseteq X$ is said to be Ψ-open if $B \subseteq \mathrm{int}_R^A(\Psi_A(B))$. The complement of a Ψ-open set is said to be Ψ-closed.

(ii) $B \subseteq X$ is said to be preopen if $B \subseteq \mathrm{int}_R^A(\mathrm{cl}_R^A(B))$. The complement of a preopen set is said to be preclosed.

(iii) $B \subseteq X$ is said to be $\mathcal{G}$-preopen if $B \subseteq \mathrm{int}_R^A(\mathrm{cl}_\Psi^A(B))$. The complement of a $\mathcal{G}$-preopen set is said to be $\mathcal{G}$-preclosed.

Lemma 3.4. *Let $(X, R, \mathcal{G})$ be a grill approximation space associated with $A \in 2^X$. Then*

(i) $B \subseteq X$ is Ψ-closed if $B \supseteq \Psi_A(\mathrm{int}_R^A(B))$.

(ii) $B \subseteq X$ is preclosed if $B \supseteq \mathrm{cl}_R^A(\mathrm{int}_R^A(B))$.

(iii) $B \subseteq X$ is $\mathcal{G}$-preclosed if $B \supseteq \mathrm{cl}_R^A(\mathrm{int}_\Psi^A(B))$.

Proof. (i) Suppose that B is Ψ-closed, then we have

$$B^c \subseteq \mathrm{int}_R^A(\Psi_A(B^c)) \subseteq \mathrm{int}_R^A(\mathrm{cl}_R^A(B^c)) = (\mathrm{cl}_R^A(\mathrm{int}_R^A(B)))^c \subseteq (\Psi_A(\mathrm{int}_R^A(B)))^c.$$

Therefore, $\Psi_A(\mathrm{int}_R^A(B)) \subseteq B$. □

Clearly, Ψ-open (Ψ-closed) $\Rightarrow$ preopen (preclosed) $\Rightarrow$ $\mathcal{G}$-preopen ($\mathcal{G}$-preclosed).

Example 3.5. Let $X = \{a, b, c, d, e\}$, $X|R = \{\{a, b\}, \{c, d\}, \{e\}\}$ and $A = \{a, b, c\}$. Then, $L_R(A) = \{a, b\}, U_R(A) = \{a, b, c, d\}$ and $(L_R(A))^c = \{c, d, e\}$.

Define a grill $\mathcal{G}$ on X as follows: $\mathcal{G} = 2^X - \{\phi, \{a\}, \{c\}, \{a, c\}\}$. Only the subsets $\{c, d, e\}, \phi, X$ satisfy $\mathrm{cl}_R^A(H) = H$, and then $\Psi_A(U_R(A)) = X$.

For $B = \{c, d\}$, then $U_R(B) = \{c, d\}$, $\mathrm{cl}_R^A(B) = \{c, d, e\}$ and then $\mathrm{cl}_\Psi^A(B) = \mathrm{cl}_R^A(B) \cup \Psi_A(U_R(A)) = X$. Hence, $B \subseteq \mathrm{int}_R^A(\mathrm{cl}_\Psi^A(B)) = \mathrm{int}_R^A(X) = X$, and thus B is a $\mathcal{G}$-preopen set. But $B \nsubseteq \mathrm{int}_R^A(\mathrm{cl}_R^A(B)) = \phi$, and then B is not a preopen set.

EXAMPLE 3.6. Let $X = \{a, b, c\}$, $X|R = \{\{a\}, \{b, c\}\}$ and define a grill $\mathcal{G}$ as follows $\mathcal{G} = \{\{c\}, \{a, c\}, \{b, c\}, X\}$, $A = \{a, b, d, e\}$. Then, $L_R(A) = \{a\}, U_R(A) = \{a, c\}$ and $(L_R(A))^c = \{b, c\}$.

For $B = \{a, c\}$, we have $U_R(B) = X$, and then $\mathrm{cl}_R^A(B) = X$ and so $\Psi_A(U_R(A)) = \Psi_A(X) = \{b, c\}$. That is, $\mathrm{cl}_\Psi^A(B) = \mathrm{cl}_R^A(B) \cup \Psi_A(U_R(A)) = X$, and $B \subseteq \mathrm{int}_R^A(\mathrm{cl}_\Psi^A(B)) = X$, which means that B is a $\mathcal{G}$-preopen set, and moreover B is preopen. But $\Psi_A(B) = \{b, c\}$ and thus $B \not\subseteq \mathrm{int}_R^A(\Psi_A(B)) = L_R(\Psi_A(B)) \cap L_R(A) = \{b, c\} \cap \{a\} = \phi$. Hence, B is not a Ψ-open set.

COROLLARY 3.7. *Let (X, R) be an approximation space associated with $A \in 2^X$. Then, $B \subseteq X$ is Ψ-open (resp. preopen or $\mathcal{G}$-preopen iff B is Φ-open (resp. preopen or ℓ-preopen) in (X, R, ℓ) where $\ell = 2^X - \mathcal{G}$.*

B is Φ-open if $B \subseteq \mathrm{int}_R^A(\Phi_A(B))$ and the complement of Φ-open is Φ-closed. B is ℓ-preopen if $B \subseteq \mathrm{int}_R^A(\mathrm{cl}_\Phi^A(B))$ and the complement of $\mathcal{G}$-preopen is ℓ-preclosed.

DEFINITION 3.8. Let $(X, R, \mathcal{G})$ be a grill approximation space associated with $A \in 2^X$. Then,
(i) A grill approximation space $(X, R, \mathcal{G})$ is said to be grill-T_0 if for every $x \neq y \in A$, there exists $G \subseteq X$ with $x \in \mathrm{int}_\Psi^A(G)$ such that $y \notin G$ or there exists $H \subseteq X$ with $y \in \mathrm{int}_\Psi^A(H)$ such that $x \notin H$.

(ii) A grill approximation space $(X, R, \mathcal{G})$ is said to be grill-T_1 if for every $x \neq y \in A$, there exist $G, H \subseteq X$ with $x \in \mathrm{int}_\Psi^A(G)$, $y \in \mathrm{int}_\Phi^A(H)$ such that $y \notin G$ and $x \notin H$.

(iii) A grill approximation space $(X, R, \mathcal{G})$ is said to be grill-T_2 if for every $x \neq y \in A$, there exist $G, H \subseteq X$ with $x \in \mathrm{int}_\Psi^A(G)$, $y \in \mathrm{int}_\Psi^A(H)$ such that $G \cap H = \phi$.

DEFINITION 3.9. Let $(X, R, \mathcal{G})$ be a grill approximation space associated with $A \in 2^X$. Then,
(i) Two sets $B, C \in 2^X$ are said to be grill approximation separated if $\mathrm{cl}_\Psi^A(B) \cap C = B \cap \mathrm{cl}_\Psi^A(C) = \phi$.

(ii) A set $G \in 2^X$ is said to be grill approximation disconnected if there exist grill approximation separated sets $B, C \in 2^X$ such that $B \cup C = G$. A set G is said to be grill approximation connected (grill App-conn.) if it is not grill approximation disconnected (grill App-disconn.).

(iii) $(X, R, \mathcal{G})$ is called a grill approximation disconnected space if there exist grill approximation separated sets $B, C \in 2^X$, such that $B \cup C = X$. A grill approximation space$(X, R, \mathcal{G})$ is called grill approximation connected if it is not grill approximation disconnected.

REMARK 3.10. Any two grill approximation separated sets $B, C \subseteq X$ are separated approximation sets as well (from that $\mathrm{cl}_R^A(G) \subseteq \mathrm{cl}_\Psi^A(G)$, $\forall G \in 2^X$). That is, grill approximation disconnectedness implies approximation disconnectedness, and thus approximation connectedness implies grill approximation connectedness.

COROLLARY 3.11. *$(X, R, \mathcal{G})$ with respect to $A \in 2^X$ is a grill-T_i approximation space ($i = 0, 1, 2$) and grill approximation connected iff (X, R, ℓ) with respect to $A \in 2^X$ is* *ideal-T_i approximation space ($i = 0, 1, 2$) and ideal approximation connected, respectively. Take $\ell = 2^X - \mathcal{G}$, $\mathcal{G} = 2^X - \ell$.*

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