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BI-CLOSING WORDS

Somayyeh Jangjooye Shaldehi

Abstract. We will show that factor codes that have a bi-closing word are bi-closing a.e. and have a degree. Moreover, a closing code with a bi-closing word into an irreducible shift space is constant-to-one. Then we study which properties may be invariant by codes that have a bi-closing word under factoring and extension. Moreover, we give some conditions for a bi-closing word and show that every closing open code in an irreducible shift space has a bi-closing word.

1. Introduction

The study of constant-to-one extensions of shift spaces is of interest in Symbolic Dynamics. In particular, the search for preserved properties by such extensions has a long history. One such property is the 'coded' property, defined by Blanchard and Hansel [1] as a generalization of irreducible sofic subshifts. Coded systems are the closure of the set of sequences obtained by free concatenation of the words in a set of words. Blanchard in [2] defined the notation of the bi-closing word and used it to prove that every irreducible constant-to-one extension of a coded system is coded. The purpose of this note is to determine some properties of codes with such a word and to give some conditions for a factor code with a bi-closing word.

A summary of our results is as follows. In Section 3 we give some properties of codes that have a bi-closing word. First, we show in Theorem 3.4 that such codes, if they consist of a maximal fiber ℓ on an irreducible subshift, have degree ℓ and are also a.e. bi-closing. Moreover, a closing code with a bi-closing word into an irreducible shift space is constant-to-one (Theorem 3.6). Theorems 3.7 and 3.9 then study the obtained properties of codes with a bi-closing word under extension and factoring, respectively. Corollary 3.11 also shows that X is of finite type if an irreducible cover of a sofic shift X has a bi-closing word.

In Section 4 we give some conditions to have a bi-closing word. The main result in this section is that every closing open code into an irreducible shift space has a bi-closing word (Theorem 4.1). Finally, in Section 5 it is proved that double transitivity

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is a totally invariant property for codes with a bi-closing word (Corollary 5.3), and as a result Theorem 5.4 shows that any factor code with a bi-closing word on a synchronized system is hyperbolic.

2. Background and notations

This section is devoted to the basic definitions and theories of symbolic dynamics [12]. Let $\mathcal{A}$ be a nonempty finite set and $\mathcal{A}^{\mathbb{Z}}$ be the collection of all bi-infinite sequences of symbols from $\mathcal{A}$. The *shift map* $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by $(\sigma x)_i = x_{i+1}$. The pair $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ becomes a dynamical system called the *full shift.* A *shift space* (or *subshift*) is a closed σ-invariant subset of $\mathcal{A}^{\mathbb{Z}}$.

Let $\mathcal{B}_n(X)$ be the set of all words of length n occurring in points in X and $\mathcal{B}(X) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(X)$. For $u \in \mathcal{B}(X)$ the set ${}_l[u] = \{x \in X : x_{[l,l+|u|-1]} = u\}$ is a *cylinder.* A subshift X is *irreducible* if for every pair of words $u, v \in \mathcal{B}(X)$ there is a word $w \in \mathcal{B}(X)$ with $uwv \in \mathcal{B}(X)$. The *orbit* of $x \in X$ is given by $Orb(x) = \{\sigma^n x : n \in \mathbb{Z}\}$. The shift space X is *minimal* if $\overline{Orb(x)} = X$ for each $x \in X$.

Suppose $x = \cdots x_{-1}x_0x_1 \cdots$ is a sequence of symbols in a shift space X over $\mathcal{A}$. We can transform x into a new sequence $y = \cdots y_{-1}y_0y_1 \cdots$ over another alphabet $\mathcal{D}$ as follows. Fix the integers m and n with $-m \le n$. To compute the i-th coordinate y_i of the transformed sequence, we use a function Φ that depends on the window of coordinates of x from $i - m$ to $i + n$. Here $\Phi : \mathcal{B}_{m+n+1}(X) \to \mathcal{D}$ is a fixed block mapping, called $(m+n+1)$-*block mapping*, from the set of all words of length $m+n+1$ occurring in points in X to symbols in $\mathcal{D}$, and so

$$y_i = \Phi(x_{i-m}x_{i-m+1} \ldots x_{i+n}) = \Phi(x_{[i-m,i+n]}). \tag{1}$$

Then the mapping $\varphi : X \to \mathcal{D}^{\mathbb{Z}}$ defined by $\varphi(x) = y$ with y_i, given by (1), is called the *sliding block code* (or *code*) with *memory* m and *anticipation* n induced by Φ. We will denote the formation of φ from Φ by $\varphi = \Phi_{\infty}^{[-m,n]}$. If $m = n = 0$, then $\varphi = \Phi_{\infty}$ is a 1-*block code.* An onto-code $\varphi : X \to Y$ is a *factor code.* Then X is an *extension* of Y. If the factor code φ is invertible, then it is called a *conjugacy.*

We call a code $\varphi : X \to Y$ *finite-to-one* if the set $\{|\varphi^{-1}(y)| : y \in Y\}$ is finite. A finite-to-one code is *constant-to-one* if the points in the image have the same number of preimages.

A shift space X is called *shift of finite type (SFT)* if there exists a finite set $\mathcal{F}$ of words in $\mathcal{A}^{\mathbb{Z}}$ such that X consists of all points in which there is no occurrence of words in $\mathcal{F}$.

If G is a directed graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, then a shift space X_G whose elements are the set of all bi-infinite paths of G is called an *edge shift.* Let $\mathcal{G} = (G, \mathcal{L})$ be a labeled graph, where G is a directed graph and $\mathcal{L} : \mathcal{E} \to \mathcal{A}$ its label. Then the shift space is $X_{\mathcal{G}} = \mathcal{L}_{\infty}(X_G)$ is a *sofic shift* and $\mathcal{G}$ is a *cover* of $X_{\mathcal{G}}$. Every factor of a shift of finite type is sofic. A *strictly sofic shift* is a sofic shift which is not of finite type.

Let $\mathcal{G} = (G, \mathcal{L})$ be a labeled graph. A word $w \in \mathcal{B}(X_{\mathcal{G}})$ is called a *synchronizing* *word* for $\mathcal{G}$ if all paths with label w terminate at the same vertex. If for each vertex I of G the edges starting at I carry different labels, then $\mathcal{G}$ is *right-resolving*. A *Fischer cover* of a sofic shift space X is a right-resolving cover with the fewest vertices among all right-resolving covers of X.

A cover is *right-closing with delay* D if two paths of length $D + 1$ always start at the same vertex and have the same label, then they must have the same initial edge. *Left-closing covers* are defined similarly. A bi-closing cover is both left-closing and right-closing.

A word v in $\mathcal{B}(X)$ is a *synchronizing* word for X if whenever $uv, vw \in \mathcal{B}(X)$, then we have $uvw \in \mathcal{B}(X)$. A *synchronized system* is an irreducible shift space X with a synchronizing word.

A point x in a shift space X is called *right-transitive* if every word in $\mathcal{B}(X)$ appears in $x_{[0,+\infty)}$. *Left-transitive points* are defined similarly. A point x in a shift space X is called *doubly transitive* if it is both left- and right-transitive.

For $x \in \mathcal{B}(X)$, let $x_- = (x_i)_{i<0}$ and $x_+ = (x_i)_{i\in\mathbb{Z}^+}$. Moreover, let $X^+ = \{x_+ : x \in X\}$. Then the successor set of x_- (resp. $w \in \mathcal{B}(X)$) is defined as $\omega_+(x_-) = \{x_+ \in X^+ : x_-x_+ \in X\}$ (resp. $\omega_+(w) = \{x_+ \in X^+ : wx_+ \in X\}$). For an irreducible shift space X, if there exists a word $w \in \mathcal{B}(X)$ and a left-transitive point $x \in X$ such that $x_{[-|w|+1,0]} = w$ and $\omega_+(x_{(-\infty,0]}) = \omega_+(w)$, then X is called *half-synchronized* and w is a *half-synchronizing word* for X.

Suppose $\varphi = \Phi_\infty : X \to Y$ is a 1 block code. For $w = w_1 \cdots w_m \in \mathcal{B}_m(Y)$ and $1 \le i \le m$, we define $d^*(w, i)$ as the number of distinct symbols seen at coordinate i in the preimages of the word w. Now we set $d^* = \min\{d^*(w, i) : w \in \mathcal{B}(Y), 1 \le i \le |w|\}$. A *magic word* is such a word w that $d^*(w, i) = d^*$ for any i. Then the index i is called a *magic coordinate*.

If for a factor code $\varphi : X \to Y$ there is a positive integer d such that every doubly transitive point $y \in Y$ has exactly d preimages, then d is the *degree* of φ and φ is d-to-one a.e.

3. Properties of codes that have a bi-closing word

Recall that any code can be recoded into a 1-block code [12, Proposition 1.5.12]. So without loss of generality, we assume throughout the paper that $\varphi : X \to Y$ is a 1-block factor code; that is, $\varphi = \Phi_\infty : X \to Y$. If $\varphi : X \to Y$ is a code of maximal fiber ℓ, then $|\varphi^{-1}(y)| \le \ell$ means for all $y \in Y$ and for at least one $y \in Y$, $|\varphi^{-1}(y)| = \ell$. Given a word $v \in \mathcal{B}(Y)$, the word $u \in \mathcal{B}(X)$ is a *lift* of v if $\Phi(u) = v$. The following notion first appeared in [2].

Definition 3.1. Let $\varphi : X \to Y$ be a 1-block factor code of maximal fiber ℓ. A word $v \in \mathcal{B}(Y)$ is called *bi-closing* if there is a factorization $w'ww''$ of v and ℓ distinct lifts $u_1, u_2, \ldots, u_\ell$ of w such that for each occurrence of v in a point $y \in Y$ the corresponding lifts of w in the ℓ preimages of y are exactly $u_1, u_2, \ldots, u_\ell$.

Blanchard [2] gives the following condition for the existence of bi-closing words.

THEOREM 3.2. *Every constant-to-one factor code has a bi-closing word.*

The following example shows that a factor code with a bi-closing word can be neither bi-closing nor constant-to-one. But in Theorem 3.4 we prove that these properties hold almost everywhere.

EXAMPLE 3.3. Let X be the sofic shift in Figure 1, $Y = \{0,1\}^{\mathbb{Z}}$ and Φ be a 1-block mapping $0 \to 0$, $1 \to 1$ and $2 \to 1$ inducing $\varphi = \Phi_\infty : X \to Y$.

Given $u \in \mathcal{B}(X)$, let $v = \Phi(u)$. Then it is obvious that $\varphi({}_\ell[u]) = {}_\ell[v]$ for $\ell \geq 0$ and thus φ is open. Moreover, we have $|\varphi^{-1}(y)| = 2$ for $y \neq 0^\infty$, while $|\varphi^{-1}(0^\infty)| = 1$. In Theorem 4.5 we will prove that any finite-to-one open code on a synchronized system has a bi-closing word. Thus, since φ is finite-to-one and open, it has bi-closing words (for example, the word 1). But φ is not bi-closing, since otherwise it would be constant-to-one [10, Theorem 4.4].

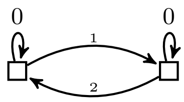

Figure 1: The Fischer cover of X.

Recall that a code $\varphi : X \to Y$ is called *right-closing almost everywhere (a.e.)* if whenever left-transitive points x and $\overline{x}$ are left-asymptotic and $\varphi(x) = \varphi(\overline{x})$, then $x = \overline{x}$. Left-closing a.e. codes and bi-closing a.e. codes are defined similarly.

THEOREM 3.4. *Let $\varphi : X \to Y$ be a factor code of maximal fiber ℓ with a bi-closing word and X be an irreducible shift space. Then φ has degree ℓ and is a.e. bi-closing.*

Proof. First we show that φ has degree ℓ. Let $v = w'ww'' \in \mathcal{B}(Y)$ be a bi-closing word, and let y be a point in Y containing v. According to Definition 3.1, there are ℓ different lifts $u_1, u_2, \ldots, u_\ell$ of w such that for each occurrence of v at the point y in Y, the corresponding lifts of w in the preimages of y are exactly $u_1, u_2, \ldots, u_\ell$. Thus y has at least ℓ preimages. On the other hand, φ is of maximal fiber ℓ. Therefore y has at most ℓ preimages and hence exactly ℓ preimages. Since v appears infinitely often to the left and to the right in any doubly transitive point, φ thus has degree ℓ.

Now suppose that φ is not right-closing a.e. Then there are various left-transitive points x and $\overline{x}$ which are left-asymptotic, and $\varphi(x) = \varphi(\overline{x}) = y$. Thus, there is an integer i for which $x_{(-\infty,i]} = \overline{x}_{(-\infty,i]}$. Since y is left-transitive, v appears infinitely often on the left in $y_{(-\infty,i]}$. We now consider the first appearance of v in $y_{(-\infty,i]}$. Without loss of generality, we can assume that u_1 is the corresponding lift of the first occurrence of w in $x_{(-\infty,i]} = \overline{x}_{(-\infty,i]}$. So $u_2, u_3, \ldots, u_\ell$ appear in the other $\ell - 1$ preimages of y. But x and $\overline{x}$ are different, so y has at least $\ell + 1$ preimages, which is a contradiction. Therefore φ is right-closing a.e. Similarly, it is left-closing a.e. □

THEOREM 3.5. *Let X be an irreducible sofic shift with the Fischer cover $\mathcal{G} = (G, \mathcal{L})$. Then $\mathcal{L}_\infty$ has a bi-closing word if and only if X is a shift of finite type.*

Proof. If X is of finite type, then $\mathcal{L}_\infty$ is a conjugacy [12]. So by Theorem 3.2 it has a bi-closing word.

For convenience, suppose that $\mathcal{L}_\infty$ has a bi-closing word. Recall from [12] that the Fischer cover of a sofic shift has a synchronizing word that appears infinitely often to the left in any doubly transitive point $x \in X$. Thus, since $\mathcal{L}_\infty$ is right-resolving, x has only one preimage, i.e., the degree of $\mathcal{L}_\infty$ is 1. Thus, by Theorem 3.4 $\mathcal{L}_\infty$ is a conjugacy and X is of finite type. □

Let X be a strictly sofic shift with Fischer cover $\mathcal{G} = (G, \mathcal{L})$. Then by Theorem 3.5, $\mathcal{L}_\infty$ has no bi-closing word by Theorem 3.5.

The main ingredient for the proof of the following theorem is given in the next section.

Theorem 3.6. *Let $\varphi : X \to Y$ be a closing factor code with a bi-closing word and Y be an irreducible shift space. Then φ is constant-to-one. In particular, if φ is bi-closing, then it is open.*

Factor codes with a bi-closing word lift the property 'coded' [2, Proposition 12]. But Fiebig gave a non-synchronized constant-to-one extension of an irreducible shift of finite type [6]. This means that the properties 'shift of finite type', 'AFT' and 'sofic' are not preserved by the extension of a code which has a bi-closing word. However, according to Theorem 3.6 and [10, Corollary 4.3], closing codes with a bi-closing word lift these properties as follows.

Theorem 3.7. *Let $\varphi : X \to Y$ be a closing code with a bi-closing word and Y be an irreducible shift of finite type (resp. strictly AFT or non-AFT sofic). Then X is of finite type (resp. strictly almost Markov or non-AFT sofic).*

As mentioned earlier, Fiebig gave a constant-to-one code $\varphi : X \to Y$ such that X is an irreducible non-synchronized shift and Y is an irreducible shift of finite type [6]. This example shows that the derived set is not obtained by a code that has a bi-closing word. For if Y is an irreducible shift of finite type, $\partial Y = \emptyset$ [15], while $\partial X \neq \emptyset$ and hence $\varphi(\partial X) \neq \partial Y$.

We now examine some properties that are invariant in codes with a bi-closing word under factoring. The following notation is motivated by the hyperbolic homeomorphism $\varphi : D(X) \to D(Y)$ defined in [16], where $D(X)$ stands for the set of doubly transitive points of X.

Definition 3.8. Let $\varphi : X \to Y$ be a factor code and X be an irreducible shift space. We call φ *hyperbolic* if there exists $d \in \mathbb{N}$ and a word $w \in \mathcal{B}_{2n+1}(Y)$ and d words $m^{(1)}, m^{(2)}, \ldots, m^{(d)} \in \mathcal{B}_{2k+1}(X)$ in which $k \leq n$, so that
(i) if $y \in Y$ is such that $y_{[-n,n]} = w$, then $\varphi^{-1}(y)_{[-k,k]} = \{m^{(1)}, m^{(2)}, \ldots, m^{(d)}\}$.

(ii) if $w' = ww''w \in \mathcal{B}(Y)$, then for every $1 \leq i \leq d$ there is a unique word $a^{(i)} \in \mathcal{B}(X)$, such that for every $x \in X$ with $x_{[-k,k]} = m^{(i)}$ and $\varphi(x)_{[-n,n+p]} = w'$, $x_{[-k,k+p]} = a^{(i)}$ holds.

Theorem 3.9. *Let $\varphi : X \to Y$ be a factor code with a bi-closing word and X be an irreducible shift of finite type (resp. synchronized or half-synchronized). Then Y is of finite type (resp. synchronized or half-synchronized).*

Proof. Since φ has bi-closing words, it is of maximal fiber d, where d is the degree of φ (Theorem 3.4). On the other hand, the degree of a finite-to-one code on an irreducible shift of finite type is the minimal number of preimages of points in Y [12, Theorem 9.1.11]. Thus, if X is of finite type, φ is constant-to-one. The result then follows from the fact that any constant-to-one factor of an irreducible shift of finite type is of finite type [10].

Now let X be synchronized. Since φ has a degree, it is hyperbolic [8, Theorem 3.3] and hence Y is synchronized [8, Theorem 4.2]. In the case of half-synchronized, the existence of the degree after the proof of [9, Theorem 3.3] leads to the result. □

As we mentioned earlier, a factor code with a bi-closing word cannot be constant-to-one (Example 3.3). But by proving Theorem 3.9 and Theorem 3.2, we have:

Theorem 3.10. *Let $\varphi : X \to Y$ be a factor code and X be an irreducible shift of finite type. Then φ has a bi-closing word if and only if it is constant-to-one.*

Note that according to Theorem 3.9 the necessity condition in Theorem 3.5 holds for all irreducible covers of sofic shifts as follows.

Corollary 3.11. *Let X be an irreducible sofic shift and $\mathcal{G} = (G, \mathcal{L})$ be an irreducible cover of X such that $\mathcal{L}_\infty$ has a bi-closing word. Then X is a shift of finite type.*

In Example 3.12 we give an irreducible right-resolving cover of the full 2-shift without bi-closing words. Therefore, sufficiency in Theorem 3.5 may fail for irreducible right-resolving covers.

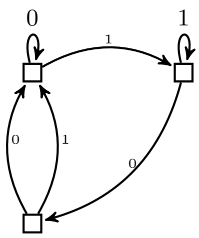

Figure 2: A right-resolving cover of full 2-shift.

Example 3.12. Let $X = \{0, 1\}^{\mathbb{Z}}$ be the full 2-shift and $\mathcal{G} = (G, \mathcal{L})$ be an irreducible right-resolving cover of X, as shown in Figure 2. For $w = 000$ we have $d^*(w, 3) = 1$ and hence $d^* = d^*(w, 3) = 1$. Recall that the degree of a finite-to-one factor code on an irreducible shift of finite type is equal to d^* [12, Theorem 9.1.11]. Therefore, we have $d = d^* = 1$. So if $\mathcal{L}_\infty$ has bi-closing words, then Theorem 3.4 implies that it is a conjugacy. But the number of preimages of $x = (01)^\infty$ is greater than 1 and this is a contradiction.

Blanchard in [2] shows that every constant-to-one factor code has a bi-closing word, but the converse is not true (Example 3.3). However, we have the following result.

THEOREM 3.13. *Let $\varphi : X \to Y$ be a factor code of maximal fiber ℓ. If there are some $k \in \mathbb{N}$ such that each word in $\mathcal{B}_k(Y)$ is bi-closing, then φ is constant-to-one and bi-closing. In particular, it is open.*

Proof. Since every word in $\mathcal{B}_k(Y)$ is bi-closing, $|\phi^{-1}(y)| \geq \ell$ for all $y \in Y$ and hence φ is constant-to-one.

Now suppose that there are distinct points x and $\overline{x}$ in X such that $x_{(-\infty,N]} = \overline{x}_{(-\infty,N]}$ and $\varphi(x) = \varphi(\overline{x}) = y$. By Definition 3.1 since$y_{[N-k+1,N]} \in \mathcal{B}_k(Y)$ is bi-closing and φ is a factor code of maximal fiber ℓ, there is a factorization $w'ww''$ of $y_{[N-k+1,N]}$ and ℓ with different lifts $u_1, u_2, \ldots, u_\ell$ of w such that the corresponding lifts of w in the preimages of y are exactly $u_1, u_2, \ldots, u_\ell$. Without loss of generality, we can assume that u_1 is the corresponding lift of w in $x_{(-\infty,N]} = \overline{x}_{(-\infty,N]}$. So $u_2, u_3, \ldots, u_\ell$ appear in the other $\ell - 1$ preimages of y. But x and $\overline{x}$ are distinct. Therefore $|\varphi^{-1}(y)| \geq \ell + 1$, which is a contradiction. So φ is right-closing. Likewise, it is left-closing. □

REMARK 3.14. The hypothesis in Theorem 3.13 does not hold for all constant-to-one codes. For then φ is bi-closing and thus open [10, Proposition 4.5]; while there is a constant-to-one code which is not open [10, Example 5.3]. Also, not all bi-closing codes satisfy the hypothesis in Theorem 3.13. For example, let X be the even shift with its Fischer cover $\mathcal{G} = (G, \mathcal{L})$ in Figure 3. Note that $\mathcal{L}_\infty$ is bi-resolving, but is neither open nor constant-to-one. So $\mathcal{L}_\infty$ has no bi-closing words.

4. Conditions for having a bi-closing word

As we have already mentioned previously, Blanchard has shown that constant-to-one factor codes have a bi-closing word (Theorem 3.2). Now we give further sufficient conditions for a factor code to have a bi-closing word.

THEOREM 4.1. *Every closing open code into an irreducible shift space has a bi-closing word.*

Proof. Let $\varphi : X \to Y$ be a closing open code and Y be irreducible. Without loss of generality, we assume that φ is right-closing. First, note that any right-closing code is finite-to-one [12, Proposition 8.1.11]. Since φ is open, it has a degree, say d, and $|\varphi^{-1}(y)| \leq d$ for all $y \in Y$ [10, Lemma 2.5]. We show that the number of preimages of y is at least d. Thus φ is constant-to-one and hence has a bi-closing word [2].

Recode φ to a right-resolving 1-block code and let $y \in Y$. Then for each n there are at least d^* distinct symbols at coordinate 0 in the preimages of $y_{[-n,n]}$. So by compactness y has at least d^* preimages.

Now suppose that $y \in Y$ is a double transitive point and $w \in \mathcal{B}(Y)$ is a magic word. Then w appears in y infinitely often to the left and to the right. Since φ is right-resolving, we have $|\varphi^{-1}(y)| \leq d^*$. Thus, $|\varphi^{-1}(y)| \leq d^*$ for every doubly transitive point $y \in Y$. On the other hand, according to the previous paragraph, $|\varphi^{-1}(y)| \geq d^*$ for all $y \in Y$. Therefore, $|\varphi^{-1}(y)| = d^*$ for every doubly transitive point $y \in Y$. Since the number of preimages of any doubly transitive point in Y is d, so $d = d^*$ and hence by the previous paragraph the number of preimages of any point in Y is at least d. □

Remark 4.2. Note that a closing code or open code may not contain bi-closing words. For example, $\mathcal{L}_\infty$ in Remark 3.14 has a bi-closing cover, but it has no bi-closing words. In the case where φ is open, let X be an irreducible non-coded shift and $x = 0^\infty$. Define $\varphi : X \to \{x\}$ to be the constant function. Since any irreducible extension of a coded system by a code that has a bi-closing word is coded [2, Proposition 12], φ has no bi-closing words.

Moreover, a code with a bi-closing word may be neither closing nor open. Jung gives such a code [10, Example 5.3].

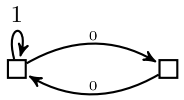

Figure 3: The Fischer cover of the even shift.

We are now ready to prove Theorem 3.6.

Proof. By the last two paragraphs of the proof of Theorem 4.1, φ has degree $d = d^*$ and $|\varphi^{-1}(y)| \geq d$ for all $y \in Y$. On the other hand, since φ has a bi-closing word, Theorem 3.4 implies that the number of preimages of any point in Y is at most d. So φ is constant-to-one. The second part is a direct application of [10, Theorem 4.4]. □

Theorem 4.3. *Let $\varphi : X \to Y$ be a closing factor code of maximal fiber ℓ and Y be an irreducible shift space. Then, φ has a bi-closing word if and only if it has degree ℓ.*

Proof. Theorem 3.4 gives us the necessity. If φ has degree ℓ, the proof of Theorem 4.1 implies that $|\varphi^{-1}(y)| \geq \ell$ for every $y \in Y$. So φ is constant-to-one and has a bi-closing word. □

Note that if the hyperbolic mapping φ is of maximal fiber d, then w in Defintion 3.8 is a bi-closing word for φ. Fiebig [8] showed that the factor code φ of a synchronized system is hyperbolic if it has degree. Thus, we have the following.

Theorem 4.4. *Let $\varphi : X \to Y$ be a factor code with degree d and X be a synchronized system. If φ is of maximal fiber d, then it has a bi-closing word.*

Theorem 4.5. *Let $\varphi : X \to Y$ be a finite-to-one open code and X be a shift space that is either synchronized or minimal. Then φ has a bi-closing word.*

Proof. Let us first assume that X is synchronized. Since φ is open and finite-to-one, it has a degree, say d, and $|\varphi^{-1}(y)| \leq d$ for all $y \in Y$ [10, Lemma 2.5]. Then by Theorem 4.4, we have that X is synchronized implies that φ has a bi-closing word.

Now let X be minimal. Recall that a finite-to-one open factor code on a minimal system is constant-to-one [13, Lemma 4.1]. Thus, by Theorem 3.2, φ has a bi-closing word. □

5. Relatively minimality

DEFINITION 5.1. Let $\varphi : X \to Y$ be a factor code between shift spaces. Then X is called *relatively minimal to* (Y, φ) if the only subshift $U \subseteq X$ for which $\varphi(U) = Y$ is $U = X$.

Given a factor code $\varphi : X \to Y$, we have $D(X) \subseteq \varphi^{-1}(D(Y))$. The following theorem shows that relative minimality is equivalent to $\varphi^{-1}(D(Y)) = D(X)$.

THEOREM 5.2. *Let $\varphi : X \to Y$ be a factor code and X be an irreducible shift space. Then X is relatively minimal to (Y, φ) if and only if $\varphi^{-1}(D(Y)) = D(X)$.*

Proof. Suppose first that $\varphi^{-1}(D(Y)) = D(X)$. Let Z be a subshift of X such that $\varphi(Z) = Y$ and y is a doubly transitive point in Y. Then there is a double transitive point $z \in Z$ such that $\varphi(z) = y$. Since the closure of the orbit of z is dense in X, we have $Z = X$.

For convenience, we assume that there is a doubly transitive point $\varphi(x)$ such that x is not doubly transitive. Then there is a word $w \in \mathcal{B}(X)$ that does not occur infinitely often to the right of x. Let $\mathcal{B}(Z) = \mathcal{B}(X) \setminus \{w\}$ be the language of the subshift Z of X.

Any word $v \in \mathcal{B}(Y)$ occurs infinitely often on the right in $\varphi(x)$. Thus, there is a preimage u of v that occurs infinitely often on the right in x. By compactness there is a limit point $z \in X$ of $\{\sigma^n(x) : n \geq 0\}$ which contains u. But w cannot appear in z and thus $z \in Z$. Thus v is a word in $\varphi(Z)$ and v is arbitrary, which means that $\varphi(Z) = Y$. This contradicts the fact that X is relatively minimal. □

Recall that if X is irreducible and $\varphi : X \to Y$ is a factor code with a bi-closing word, then X is relatively minimal to (Y, φ) [2, Proposition 10]. Thus, by Theorem 5.2 we have:

COROLLARY 5.3. *Let $\varphi : X \to Y$ be a factor code with a bi-closing word and X be irreducible. Then $\varphi^{-1}(D(Y)) = D(X)$.*

However, the converse is not true. For example, let $\varphi : X \to Y$ be a factor code of degree 1 that is not one-to-one, and let X be irreducible. Since φ has degree, we have $\varphi^{-1}(D(Y)) = D(X)$ [8, Theorem 3.2]. But Theorem 3.4 implies that φ cannot have bi-closing words.

Note that a hyperbolic factor code cannot have bi-closing words. For example, let X be a strictly sofic shift with the Fischer cover $\mathcal{G} = (G, \mathcal{L})$. By the proof of

Theorem 3.5, $\mathcal{L}_\infty$ has degree 1. So it is hyperbolic. But $\mathcal{L}_\infty$ cannot have bi-closing words; for otherwise Theorem 3.4 implies that it is a conjugacy and thus X is of finite type.

Also, a code with a bi-closing word cannot be hyperbolic; as an example, Fiebig gives a code φ with constant factor from a non-synchronized shift to a finite type shift [6]. Since hyperbolic maps lift the 'synchronized' property [8, Theorem 4.2], φ is not hyperbolic. But by Theorem 5.3 and [8, Theorem 3.3], which states that if X is synchronized, hyperbolicity is equivalent to $\varphi^{-1}(D(Y)) = D(X)$, we have the following.

THEOREM 5.4. *Let $\varphi : X \to Y$ be a factor code with a bi-closing word and X be a synchronized system. Then φ is hyperbolic.*

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Department of Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, Iran

E-mail: s.jangjoo@alzahra.ac.ir