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ON THE LOCAL CONTROLLABILITY FOR OPTIMAL CONTROL PROBLEMS

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Abstract. We consider an optimal control problem on the fixed interval of time with the right endpoint constraint. We introduce the concept of controllability for this problem. The main result of the paper states that if for the optimal control problem the Pontryagin maximum principle fails on the given admissible process then this process satisfies the controllability condition.

1. Introduction

The paper is dedicated to the following question: how is the local extremality of a control related to the controllability in the neighbourhood of this control? We will illustrate this question with the following elementary example.

Given positive integers s and k such that $s \geq k$, a continuously differentiable function $f : \mathbb{R}^s \to \mathbb{R}$ and a continuously differentiable mapping $F : \mathbb{R}^s \to \mathbb{R}^k$, consider the optimization problem

$$f(x) \to \text{extr}, \quad F(x) = 0. \tag{1}$$

Let us assume that a given point $\overline{x} \in \mathbb{R}^s$ is admissible, i.e. $F(\overline{x}) = 0$. For the sake of simplicity, we will assume that $f(\overline{x}) = 0$ as well.

There are two cases. In the first case, the Lagrange principle fails at $\overline{x}$, i.e. $\lambda_0 f'(\overline{x}) + \lambda F'(\overline{x}) \neq 0$ for each $(\lambda_0, \lambda) \neq 0$. Then the vectors $f'(\overline{x})$, $F_1'(\overline{x})$, $\dots$, $F_k'(\overline{x})$ are linearly independent. The inverse function theorem therefore implies that there exist $\varepsilon > 0$ and const > 0, so that for each $(y_0, y) \in O_\varepsilon(0)$ the system of equations $f(x) = y_0$, $F(x) = y$ has a solution x that satisfies the inequality $|x - \bar{x}| \leq \text{const}(|y_0| + |y|)$. Here, $O_\varepsilon(0)$ is an ε-neighbourhood of zero in $\mathbb{R}^{k+1}$,

The second case: the Lagrange principle applies at the $\overline{x}$, i.e. there exists a non-zero vector $(\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^k$ such that $\lambda_0 f'(\overline{x}) + \lambda F'(\overline{x}) = 0$.

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These considerations lead us to the following conclusion. If the Lagrange principle fails for the problem (1) at $\overline{x}$, then in this point local controllability property holds, i.e. for each (y_0, y) close to zero, the system of equations $f(x) = y_0, F(x) = y$ has a solution that satisfies a priori estimate. This assertion does not provide sufficient optimality condition. It is a simple task to construct an example when $\overline{x}$ is not a solution to the problem (1) and there is no local contralability at $\overline{x}$. For example, one can take the zero mapping $F(x) \equiv 0$ and a function f for which $f'(\overline{x}) \neq 0$.

2. Formulation of the optimal control problem

Consider the control system

$$\dot{x} = f(x, u, t), \quad x(t_0) = x_0, \quad u(t) \in U(t), \quad t \in [t_0, t_1]. \tag{2}$$

Here $t \in [t_0, t_1]$ is the time, $x \in \mathbb{R}^n$ is a state variable, the left endpoint x_0 is fixed, and $u(t) \in U(t)$ for a.a. $t \in [t_0, t_1]$ is a control. The set-valued mapping $U : \mathbb{R} \rightrightarrows \mathbb{R}^s$ is given. We assume that U is measurable and essentially bounded (see e.g. [1,10]). The assumption of boundedness for $U(\cdot)$ is only necessary for simplicity of formulation.

Let us assume that the mapping $f : \mathbb{R}^n \times \mathbb{R}^s \times [t_0, t_1] \to \mathbb{R}^n$ is continuous, the mapping $f(\cdot, u, t)$ is differentiable for every u and t and $\dfrac{\partial f}{\partial x}$ is continuous. We consider the class of admissible controls consisting of all measurable, essentially bounded functions $u(\cdot)$ on $[t_0, t_1]$ such that $u(t) \in U(t)$ for a.a. $t \in [t_0, t_1]$.

Definition 2.1. A pair of functions $(x(\cdot), u(\cdot))$ is called an admissible process, if $x(t)$ is a solution to the Cauchy problem

$$\dot{x} = f(x, u(t), t), \quad x(t_0) = x_0$$

on $[t_0, t_1]$ and $u(\cdot)$ is an admissible control.

Consider the right endpoint constraint

$$\psi_1(x_1) = 0, \quad x_1 = x(t_1). \tag{3}$$

Here $\psi_1 : \mathbb{R}^n \to \mathbb{R}^{k_1}$ is a given continuously differentiable mapping and $k_1 \geq 0$ is a given non-negative integer. This condition is called the transversality condition. For the sake of generality, we can also include the dependence on $x(t_0)$ in this constraint, but we do not do so for the sake of simplicity of formulation.

If $k_1 = n$ and $\psi_1(x_1) \equiv x_1 - \bar{x}_1$, where $\bar{x}_1$ is a given vector, then we obtain a problem with the fixed right endpoint $\bar{x}_1$. If $k_1 = 0$, then we get a problem with a free right endpoint.

Let $(\widehat{x}(\cdot), \widehat{u}(\cdot))$ be an admissible process that satisfies the transversality condition (3). Set $\widehat{x}(t_1) := \widehat{x}_1$.

Consider the optimization problem for the functional

$$J(u) := \int_{t_0}^{t_1} f_0(x(t), u(t), t)dt + \psi_0(x_1) \to \text{extr} \tag{4}$$

over the set of all admissible pairs $(x(\cdot),\, u(\cdot))$ satisfying the transversality condition (3). Here, the given function f_0 satisfies the same smoothness conditions as f and a given function ψ_0 is assumed to be continuously differentiable.

We understand the locality of the optima in the sense of the metric given by the formula

$$\rho(\widehat{u}, u) = \operatorname{meas}\{t \in [t_0, t_1] : \widehat{u}(t) \neq u(t)\} \tag{5}$$

where $\widehat{u}(\cdot)$ and $u(\cdot)$ are admissible controls and meas is the Lebesgue measure. Note that this metric is complete (see e.g. [2, 8]).

Definition 2.2. We will say that the admissible process $(\widehat{x}(\cdot), \widehat{u}(\cdot))$ satisfies the maximum principle, if there exists a nonzero vector $(\lambda_0, \lambda_1) \in \mathbb{R} \times \mathbb{R}^{k_1}$ such that the transversality condition

$$p(t_1) = -\frac{\partial l}{\partial x_1}(\lambda_0, \lambda_1, \widehat{x}(t_1)) \tag{6}$$

holds and the condition of maximum of Hamiltonian with respect to u

$$H(\lambda_0, p(t), \widehat{x}(t), \widehat{u}(t), t) = \max_{u \in U(t)} H(\lambda_0, p(t), \widehat{x}(t), u, t) \quad \text{for a.a.} \quad t \in [t_0, t_1]. \tag{7}$$

holds. Here

$$l(\lambda_0, \lambda_1, x_1) = \lambda_0 \psi_0(x_1) + \langle \lambda_1, \psi_1(x_1) \rangle, \tag{8}$$

$p(t)$ – is an absolutely continuous solution to the linear with respect to p nonhomogeneous equation

$$\dot{p} = -\frac{\partial H}{\partial x}(\lambda_0, p(t), \widehat{x}(t), \widehat{u}(t), t) = -p(t)\frac{\partial f}{\partial x}(\widehat{x}(t), \widehat{u}(t), t) + \lambda_0 \frac{\partial f_0}{\partial x}(\widehat{x}(t), \widehat{u}(t), t), \tag{9}$$

the function l is the small Lagrangian. The Hamiltonian H is defined by the formula $H(\lambda_0, p, x, u, t) = -\lambda_0 f_0(x, u, t) + \langle p, f(x, u, t) \rangle$.

It is important to note that we do not assume here that $\lambda_0 \geq 0$ as in Pontryagin's maximum principle. Example 4.1 below shows that this assumption makes no sense in the context of the main result of the paper. The property formulated in Definition 2.2 can therefore be regarded as a local extremality principle, which allows both a minimum and a maximum of J at $\widehat{u}$.

Proposition 2.3. *Assume that an admissible process satisfies the maximum principle. If the regularity condition* $\operatorname{rank}\frac{\partial \psi_1}{\partial x_1}(x_1) = k_1$ *holds, then*

$$|\lambda_0| + |p(t)| \neq 0 \quad \forall\, t \in [t_0, t_1]. \tag{10}$$

Proof. Let (10) fail. Then $\lambda_0 = 0$ and $p(\tau) = 0$ for some $\tau \in [t_0, t_1]$. Due to the linearity of (9) we therefore have $p(t) = 0$ for every $t \in [t_0, t_1]$. Therefore, (6) and the regularity assumptions imply that $\lambda_0 = 0$ and $\lambda_1 = 0$, which contradicts the maximum principle. □

Definition 2.4. We will say that the admissible process $(\widehat{x}(\cdot), \widehat{u}(\cdot))$ satisfies the controllability condition, if there exists $\delta > 0$ and const > 0 such that for every $e = (e_1,\, e_0) \in \mathbb{R}^{k_1} \times \mathbb{R}$ satisfying the inequality $|e_1| + |e_0 - J(\widehat{u})| \leq \delta$ there exists an admissible process $(x(\cdot), u(\cdot))$ such that

$$\psi_1(x(t_1)) = e_1, \quad J(u) = e_0 \tag{11}$$

and

$$\rho(u, \widehat{u}) \leq \mathrm{const}\Big(|e_1| + |e_0 - J(\widehat{u})|\Big). \tag{12}$$

Here ρ is defined by formula (5).

Our aim is to show that if the admissible process $(\widehat{x}(\cdot), \widehat{u}(\cdot))$ does not satisfy the maximum principle, then it satisfies the condition of controllability.

THEOREM 2.5. *Let the admissible process* $(\widehat{x}(\cdot), \widehat{u}(\cdot))$ *does not satisfy the maximum principle, i.e. for every* $\lambda = (\lambda_0, \lambda_1) \neq 0$ *and for the corresponding solution to the adjoint system* (6) *and* (9)*, the condition of maximum of Hamiltonian* (7) *fails over a set of positive measure. Then this process satisfies the controllability condition (see Definition 2.4).*

The proof of this theorem is presented in the next section.

Theorem 2.5 cannot be strengthened by replacing its assumption with the assumption that $\widehat{u}$ is not a weak extreme point. This fact is shown in Example 4.2.

3. Proof of the main result

Since the considered optimal control problem is not autonomous, i.e. it contains the explicit dependence on t, we assume without loss of generality that $\widehat{x}(t) \equiv 0$. It follows that $x_0 = \widehat{x}(t_1) = 0$. For the sake of simplicity, let us also assume that $J(\widehat{u}) = 0$.

Let us construct a finite-dimensional approximation of the optimal control problem (see e.g. [4,5]). There exists a countable set of admissible controls $\{u_i(\cdot)\}$ such that the set $\{u_1(t), u_2(t), \ldots\}$ is everywhere dense in $U(t)$ for a.a. $t \in [t_0, t_1]$ (see e.g. [14, Ch. I, Sect. 7]). If U is not dependent on t, then one can take a countable dense subset of U and take $\{u_i(\cdot)\}$ as the corresponding constant functions.

The standard existence and uniqueness theorems (see [9, Theorem IV.2]) imply that there exists $\delta_0 > 0$ such that for every admissible control $u(\cdot)$ that satisfies the inequality $\mathrm{meas}\{t \in [t_0, t_1] : u(t) \neq \widehat{u}(t)\} \leq \delta_0$ there is the only solution to the Cauchy problem

$$\dot{x} = f(x, u(t), t), \quad x(t_0) = x_0, \quad t \in [t_0, t_1]$$

defined on the entire segment $[t_0, t_1]$.

Recall (see e.g. [12, Ch. IX, Sect. 5, 6]) that a point τ for a given function $\varphi : [t_0, t_1] \to \mathbb{R}^n$ is called an approximative continuity point if there exists a measurable set E such that the restriction of φ to E is continuous, $\varphi(t) \to \varphi(\tau)$ as $t \to \tau$, $t \in E$, and τ is a density point of the set E, i.e.

$$D_\tau E := \lim_{h \to 0} \frac{\mathrm{meas}([\tau - h, \tau + h] \cap E)}{2h} = 1 \quad \text{as} \quad h \to 0.$$

As is known, for a measurable function on $[t_0, t_1]$, almost all points $t \in [t_0, t_1]$ are the Lebesgue points and the approximative continuity points.

For every $\delta > 0$ that satisfies the inequalities $\delta < t_1 - t_0$ and $\delta < \delta_0$ there exists a compact subset $\widetilde{M}(\delta) \subset [t_0,\, t_1]$ such that the following assertions are valid. The measure of $\widetilde{M}(\delta)$ is less or equal to $(t_1 - t_0) - \delta$, all functions $\widehat{u}$ and u_i are continuous on $\widetilde{M}(\delta)$, the mapping f is continuous on $\mathbb{R}^n \times \mathbb{R}^s \times \widetilde{M}(\delta)$ (see [4, Lemma П.2.5]), all points of $\widetilde{M}(\delta)$ are the Lebesgue points of the functions $t \mapsto f(\widehat{x}(t), \widehat{u}(t), t)$ and $t \to f(\widehat{x}(t), u_i(t), t)$. The same assertions apply to f_0 and $\widetilde{M}(\delta)$. It follows from the Cantor-Bendixson theorem (see e.g. [12, Ch. II, Sect. 6]) that from the set $\widetilde{M}(\delta)$, a countable subset can be excluded, so that the set becomes perfect, i.e. it is closed and consists only of its limit points. Let us denote this perfect set by $M(\delta)$. Without loss of generality, we assume that the set-valued mapping is decreasing, i.e. $\delta > \delta' \Rightarrow M(\delta) \subset M(\delta')$.

Take a dense countable subset $\{t_i\}$, $i = 2, 3 \ldots$ in $M(\delta)$ such that $t_i \neq t_0,\, t_1$. Take a positive integer $N \geq \delta^{-1}$. Then for every positive integer $j \leq N$ there exist points $t_{j,1}(N) < \ldots < t_{j,N+1}(N)$ in $M(\delta)$ such that

$$t_j \in [t_{j,1}(N), t_{j,N+1}(N)], \quad t_{j,N+1}(N) - t_{j,1}(N) \leq \frac{\delta_0}{N} \tag{13}$$

and

$$[t_{j,1}(N), t_{j,N+1}(N)] \bigcap [t_{j',1}(N), t_{j',N+1}(N)] = \emptyset \quad \forall\, j \neq j' \in \{1, \ldots, N\}.$$

These points $t_{j,1}(N)$, $\ldots$, $t_{j,N+1}(N)$ exist because $M(\delta)$ is closed and consists only of its limit points. Here δ_0 is chosen above.

Denote by U_N the set of $N \times N$- matrices with non-negative elements $\xi_{j,p}$, $j, p \in \{1, \ldots, N\}$ such that $0 \leq \xi_{i,j} \leq c(N)$. Here

$$c(N) = \frac{1}{2} \min\{t_{s,i+1}(N) - t_{s,i}(N) :\ s, i \in \{1, \ldots, N\}\} > 0.$$

Let us assign to the matrix $\xi \in U_N$ an admissible control $u(t, \xi)$ defined by the formula

$$u(t,\xi) = \begin{cases} u_p(t) & \text{if} \quad \exists\, j \in \{1, \ldots, N\} :\ t \in (t_{j,p}(N), t_{j,p}(N) + \xi_{j,p}), \quad 1 \leq p \leq N, \\ \widehat{u}(t) & \text{otherwise.} \end{cases}$$

This construction implies that

$$\text{meas}\{t \in [t_0, t_1] :\ u(t,\xi) \neq \widehat{u}(t)\} \leq N \frac{\delta_0}{N} = \delta_0$$

for every $\xi \in U_N$. There is therefore the only mapping $X(t, \xi)$ that assigns to each matrix $\xi \in U_N$ the value at point t of the solution of the Cauchy problem

$$\dot{x} = f(x, u(t,\xi), t), \quad x(t_0) = x_0, \quad t \in [t_0, t_1], \quad 0 \leq \xi_{j,p} \leq \delta(N).$$

Set $X(\xi) = X(t_1, \xi)$. Also note that $X(t, 0) \equiv 0$ due to the assumption made above that $\widehat{x}(t) \equiv 0$, $t \in [t_0, t_1]$.

It is known that this mapping is continuous in a neighbourhood of zero and differentiable at zero with respect to the set U_N (see [4, pp. 73, 85–86]).

Set $k = k_1 + 1$. Define the mapping $F_N : U_N \to \mathbb{R}^k$ by the formula

$$F_N(\xi) = (\psi_1(X(\xi)), J(u(\cdot))).$$

A point $\xi = 0$ corresponds to the given control $\widehat{u}$ and satisfies the identity $F_N(0) = (\psi_1(\widehat{x}_1), J(\widehat{u})) = 0$.

Two cases can occur. In the first case, for some N and for the mapping F_N the Robinson condition $0 \in \mathrm{int} F_N'(0)U_N$ holds. Then it follows from Robinson's theorem (see e.g. [13]) that by decreasing $\delta > 0$ we obtain that there exists const > 0 for each $e = (e_1, e_0) \in \mathbb{R}^{k_1} \times \mathbb{R}^1$ which satisfies the inequality $|e| \le \delta$ the equation $F_N(\xi) = e$ has a solution ξ that satisfies the inequality $|\xi| \le \mathrm{const}|\varepsilon|$. Here $|\xi| = \mathrm{meas}\{t \in [t_0, t_1] : u(t, \xi) \neq \widehat{u}(t)\}$.

Note that the smoothness assumption of Robinson's theorem applies here for F_N at zero (for more details see [4]).

The second case: $0 \in \mathrm{br} F_N'(0)U_N$ for every sufficiently large N. Here br stands for the boundary of a set.

The set of non-negative $N \times N$-matrices satisfying the inequality $0 \le \xi_{j,p} \le \delta(N)$ is convex and the variational equation is linear. Therefore, the set $F_N'(0)U_N$ is convex and has zero on its boundary. Therefore, the finite-dimensional separation theorem implies that for sufficiently large N there exists a non-zero vector $d_N \in \mathbb{R}^k$ such that

$$\lim_{\xi_{j,p} \to 0+} \langle d_N, \frac{F_N(\xi) - F_N(0)}{\xi_{j,p}} \rangle|_{\xi=0} \ge 0 \quad \forall\, j, p \in \{1, \dots, N\}.$$

So if $\xi_{j,p} > 0$ we have

$$\frac{\partial}{\partial \xi_{j,p}} \langle d_N, F_N(\xi) - F_N(0) \rangle|_{\xi=0} \ge 0 \quad \forall\, j, p \in \{1, \dots, N\}. \tag{14}$$

Let us show that the admissible process $(\widehat{x}(\cdot), \widehat{u}(\cdot))$ fulfills the maximum principle. This contradiction to the assumption of the theorem completes the proof.

Set $d_N = \lambda_N = (\lambda_{0,N}, \lambda_{1,N})$. Since $d_N \neq 0$, we obtain by normalizing the vector λ_N that $|\lambda_N| = |\lambda_{0,N}| + |\lambda_{1,N}| = 1$. For an infinite number N, which we only consider for the first coordinate, $\lambda_{0,N}$ is sign-definite. For simplicity, let us assume that it is non-negative. We want to show that the process $(\widehat{x}(\cdot), \widehat{u}(\cdot))$ satisfies the maximum principle with $\lambda_0 \ge 0$.

In fact, in the transition to a subsequence, we will assume that $\lambda_{0,N} \to \lambda_0 \ge 0$, $\lambda_{1,N} \to \lambda_1$ and $\lambda = (\lambda_0, \lambda_1)$, $|\lambda| = \lambda_0 + |\lambda_1| = 1$. Let us now consider (14).

Assuming that N is fixed and the matrix ξ is such that $\xi_{j,p}$ is its only positive component, we use that F_N is differentiable at zero and its derivative can be found using the fundamental matrix of the system $\dot{y} = \frac{\partial f}{\partial x}(\widehat{x}(t), \widehat{u}(t), t)y$, $t \in [t_0, t_1]$ (see e.g. [11]).

From (14) it follows that

$$\frac{\partial}{\partial \xi_{j,p}} \langle \lambda_{1,N}, \psi_1(X(\xi)) \rangle |_{\xi=0}$$

$$+\lambda_{0,N} \frac{\partial}{\partial \xi_{j,p}} \left(\int_{t_0}^{t_1} f_0(t, \xi) dt + (\psi_0(X(\xi)) - \psi_0(X(0))) \right) |_{\xi=0} \ge 0 \qquad \forall\, j, p \in \{1, \dots, N\},$$

where $f_0(t, \xi) = f_0(X(t, \xi), u(\xi, t), t) - f_0(\widehat{x}(t), \widehat{u}(t), t)$. Here we use the identities $\widehat{x} = 0$, $\psi_1(X(0)) = 0$ and $J(\widehat{u}) = 0$.

This inequality and the definition of the small Lagrangian l imply

$$\frac{\partial}{\partial \xi_{j,p}} l(\lambda_{0,N}, \lambda_{1,N}, X(\xi))|_{\xi=0} + \lambda_{0,N} \frac{\partial}{\partial \xi_{j,p}} \left(\int_{t_0}^{t_1} f_0(t,\xi) dt \right) |_{\xi=0} \geq 0, \ \forall\, j,p \in \{1,\dots,N\}.$$

Denote by p_N the solution of the equation

$$\dot{p}_N = -p_N(t) \frac{\partial f}{\partial x}(\widehat{x}(t), \widehat{u}(t), t) + \lambda_{0,N} \frac{\partial f_0}{\partial x}(\widehat{x}(t), \widehat{u}(t), t)$$

on $[t_0,\, t_1]$ with the endpoint constraint $p_N(t_1) = -\frac{\partial l}{\partial x_1}(\lambda_{0,N}, \lambda_{1,N}, \widehat{x}(t_1))$. This problem has the only solution.

The inequality obtained, the formula for p_N and the formula for the derivative in $\xi_{j,p}$ imply

$$\xi_{j,p}^{-1} \left(-\langle p_N(t_1), X(\xi) \rangle + \lambda_{0,N} \int_{t_0}^{t_1} f_0(t,\xi) dt \right) + \kappa(\xi) \geq 0.$$

Here and in the following $\kappa(\xi) \to 0$ as $\xi j, p \to 0+$. The Newton-Leibniz formula and the relationship $X(t_0, \xi) \equiv 0$ imply

$$\xi_{j,p}^{-1} \int_{t_0}^{t_1} \left(-\frac{d}{dt} \langle p_N(t), X(t,\xi) \rangle + \lambda_{0,N} f_0(t,\xi) \right) dt + \kappa(\xi) \geq 0.$$

Since $X(t,0) \equiv 0$, we have

$$\int_{t_0}^{t_1} \langle H_x(\lambda_{0,N}, p_N(t), \widehat{x}(t), u(t,\xi), t), X(t,\xi) \rangle \, dt$$
$$= \int_{t_0}^{t_1} \langle H_x(\lambda_{0,N}, p_N(t), \widehat{x}(t), \widehat{u}(t), t), X(t,\xi) \rangle \, dt + o(\xi_{j,p}).$$

The last inequality implies

$$\int_{t_0}^{t_1} \Bigg(\left\langle -\frac{d}{dt}(p_N(t)), X(t,\xi) \right\rangle - \langle H_x(\lambda_{0,N}, p_N(t), \widehat{x}(t), \widehat{u}(t), t), X(t,\xi) \rangle$$
$$-H(\lambda_{0,N}, p_N(t), \widehat{x}(t), u(t,\xi), t) + H(\lambda_{0,N}, p_N(t), \widehat{x}(t), \widehat{u}(t), t) \Bigg) dt + o(\xi_{j,p}) \geq 0.$$

Using (9) we therefore obtain

$$\int_{t_0}^{t_1} \Big(H(\lambda_{0,N}, p_N(t), \widehat{x}(t), \widehat{u}(t), t) - H(\lambda_{0,N}, p_N(t), \widehat{x}(t), u(t,\xi), t) \Big) dt + o(\xi_{j,p}) \geq 0.$$

The point $t_{j,p}(N)$ is the Lebesgue point of the functions

$$H_p(t) = H(\lambda_{0,N}, p_N(t), \widehat{x}(t), u_p(t), t),$$
$$H(t) = H(\lambda_{0,N}, p_N(t), \widehat{x}(t), \widehat{u}(t), t)$$

for each N and $j,p \in \{1,\dots,N\}$. So, the obtained inequality and the definition of the control $u(t,\xi)$ imply that

$$\int_{t_0}^{t_1} \Big(H(\lambda_{0,N}, p_N(t), \widehat{x}(t), \widehat{u}(t), t) - H(\lambda_{0,N}, p_N(t), \widehat{x}(t), u(t,\xi), t) \Big) dt + o(\xi_{j,p})$$
$$= \xi_{j,p} (H(\lambda_{0,N}, p_N(t_{j,p}(N)), \widehat{x}(t_{j,p}(N)), \widehat{u}(t_{j,p}(N)), t_{j,p}(N))$$

$$- H(\lambda_{0,N}, p_N(t_{j,p}(N)), \widehat{x}(t_{j,p}(N)), u_p(t_{j,p}(N)), t_{j,p}(N))) + o(\xi_{j,p}) \geq 0.$$

Dividing this inequality by $\xi_{j,p} > 0$ and passing to the limits as $\xi_{j,p} \to 0$, we obtain

$$H(\lambda_{0,N}, p_N(t_{j,p}(N)), \widehat{x}(t_{j,p}(N)), \widehat{u}(t_{j,p}(N)), t_{j,p}(N))$$
$$\geq H(\lambda_{0,N}, p_N(t_{j,p}(N)), \widehat{x}(t_{j,p}(N)), u_p(t_{j,p}(N)), t_{j,p}(N))$$

for each $j, p \leq N$.

Fix the indices j, p. This inequality holds for an infinite set of numbers N. The definition of $p_N(\cdot)$ implies that these functions converge uniformly to the solution of (6), (9). Furthermore, $t_{j,p}(N) \to t_j$ as $N \to \infty$. Passing to the limit as $N \to \infty$ since the corresponding functions are continuous on $M(\delta)$, we obtain that

$$H(\lambda_0, p(t_j), \widehat{x}(t_j), \widehat{u}(t_j), t_j) \geq H(\lambda_0, p(t_j), \widehat{x}(t_j), u_p(t_j), t_j)$$

for each j and p.

Note that for each j the sequence $\{u_p(t_j)\}$ is dense in $U(t_j)$. Since the functions f and f_0 are continuous in u, we obtain that

$$H(\lambda_0, p(t_j), \widehat{x}(t_j), \widehat{u}(t_j), t_j) \geq H(\lambda_0, p(t_j), \widehat{x}(t_j), u, t_j) \quad \forall\, u \in U(t_j).$$

for each $j \in \{1, 2, \ldots\}$. The sequence $\{t_j\}$ is dense in $M(\delta)$. Moving to the limit as $j \to \infty$, we obtain the condition of the maximum of the Hamiltonian (7) holds for each $t \in M(\delta)$. Due to the arbitrariness of the choice of δ and since the union of the sets $M(\delta)$ is a set of full measure in $[t_0, t_1]$, we obtain that the condition of the maximum of the Hamiltonian (7) holds for a.a. $t \in [t_0, t_1]$.

The contradiction obtained completes the considerations for the case when $\lambda_{0,N} \geq 0$ for infinitely many N. If $\lambda_{0,N} < 0$ for all sufficiently large N, the analogous considerations apply with the opposite sign.

4. Examples

The following simple example shows that the extremality assumption in Theorem 2.5 is essential.

Example 4.1. Let $n = 1,\ t_0 = 0, t_1 = 1,\ f(x, u, t) = u$, $U = [0, 1]$, $J(u) = \int_0^1 u dt$, $x_0 = 0$, x_1 is free, the transversality condition and ψ_0 are absent.

The minimum in this problem is $\widehat{u} \equiv 0$. The corresponding trajectory $\widehat{x}$ is also a zero function. It is a simple task to ensure that the equation $J(u) = -e_0$ has no solutions for every $e_0 > 0$.

If we consider the same control system and take the functional J with the opposite sign $J(u) = -\int_0^1 u dt$, then we obtain that the maximum is reached at $\widehat{u} \equiv 0$. This provides an answer when the maximum is studied for the problem under consideration.

The following example shows that the assumptions of the theorem cannot be strengthened as follows. The assumption that the admissible process does not satisfy the maximum principle cannot be replaced by the assumption that either the assumption of the theorem holds or the admissible process is not a weak minimum. In the following example, the maximum principle holds and the admissible process is not a weak minimum, however the condition of local controllability is not fulfilled.

EXAMPLE 4.2. Let $n = 1$, $t_0 = 0$, $t_1 = 2\pi$, $f(x, u, t) = u$, $U = [-1, 1]$, $J(u) = \int_0^{2\pi}(u^2 - x^2)dt$, $x_0 = 0$, $x_1 = 0$, i.e. the two end points $x_0 = 0$ and $x_1 = 0$ are fixed and the functional ψ_0 is absent. This is a well-known problem of calculus of variations.

Pontryagin's maximum principle applies to the admissible process $(0, 0)$, since $\lambda_0 = 1$, $\lambda_1 = 0$ and $p(t) \equiv 0$. At the same time, since $2\pi > \pi$, the zero trajectory on $[0, 2\pi)$ has an adjoint point (see [10]). Therefore, there exists a sequence of admissible controls $\{u_i\}$ such that $|u_i(t)| \leq 1/i \ \forall\, t \in [0, 2\pi]$, the corresponding process (u_i, x_i) is admissible and $J(u_i) < 0$. The zero admissible process is therefore not a weak local minimum. It is also not a weak local maximum. In fact, $-J$ does not satisfy necessary minimum condition, since the Legandre condition for $\widehat{u} = 0$ fails. So $\widehat{u} = 0$ is not a weak local extreme point.

We will say that the admissible process $(\widehat{x}, \widehat{u})$ is a Pontryagin's minimum if for every positive integer $k > \operatorname{ess\,sup}_{t\in[t_0,t_1]} |\widehat{u}(t)|$ there exists $\varepsilon = \varepsilon(k) > 0$, so that for every admissible process $(x(\cdot), u(\cdot))$ that satisfies the inequalities $\rho(\widehat{u}, u) \leq \varepsilon$ and $|u(t)| \leq k$, it holds that $J(\widehat{u}) \leq J(u)$.

We want to show that for the zero process the local controllability condition fails. In fact, it satisfies the sufficient conditions of Pontryagin's minimum form [4, p. 85–86]. Therefore there exists $\delta > 0$ such that $J(0) \leq J(u)$ for each admissible $(x(\cdot), u(\cdot))$ for which $\rho(u, 0) < \delta$. Therefore, $(0, 0)$ is the Pontryagin minimum for the problem under consideration. Therefore, for all sufficiently small $\delta > 0$ the equation $J(u) = -\delta$ has no solution u such that $\rho(u, 0) < \delta$.

5. Conclusion

We have considered the optimal control problem (2), (4). For this problem, it is shown that if an admissible process $(\widehat{x}(\cdot), \widehat{u}(\cdot))$ does not satisfy the maximum principle, then this process satisfies the controllability condition.

The obtained assertion can be weakened by excluding the assumption of essential boundedness of the set-valued mapping U (see e.g. [14, I.7] or [7, 1.5.1]). In fact, take any positive integer $k_0 > \operatorname{ess\,sup}_{t\in[t_0,\, t_1]} |\widehat{u}(t)|$. For each positive integer i, the set $\{u : u \in U(t), u \in B_{k_0+i}\}$ is essentially bounded in $t \in [t_0, t_1]$ (here B_R is a closed ball centered at zero with radius R). If we apply Theorem 2.5 to the pair $(\widehat{x}, \widehat{u})$ and move on to the limit as $i \to \infty$, we conclude the considerations.

Another approach to controllability was proposed in [3, 6]. It is based on the second-order conditions for optimal control problems.

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