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EMBEDDINGS TO RECTILINEAR SPACE AND GROMOV–HAUSDORFF DISTANCES

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Abstract. We show that the problem whether a given finite metric space can be embedded into m-dimensional rectilinear space can be reformulated in terms of the Gromov–Hausdorff distance between some special finite metric spaces.

1. Introduction

We continue our investigations [25,27,28,39] on connections between the geometry of the Gromov–Hausdorff distance and Discrete Geometry problems such as calculation of edges weights of a minimal spanning tree, the Borsuk conjecture, computing the chromatic number and the clique cover number for a simple graph, etc. In this paper we give an answer to the question whether a given finite metric space (X, d) can be embedded in an m-dimensional rectilinear space, using the Gromov–Hausdorff distance.

Hyperspaces

For subsets A and B of a fixed metric space X, a natural distance function d_H was defined by F. Hausdorff [18] as the infimum of the positive numbers r, so that A is contained in the r-neighborhood of B and vice versa. It is known that this function, called *Hausdorff distance*, is a metric on the family of all closed bounded subsets of the metric space X, see for example [9]. The Hausdorff distance was generalized by D. Edwards [13] and independently by M. Gromov [15] to the case of two metric spaces X and Y. The *Gromov–Hausdorff distance* between the spaces X and Y is equal to the infimum of the values $d_H\big(\varphi(X), \psi(Y)\big)$ over all possible isometric embeddings $\varphi\colon X \to Z$ and $\psi\colon Y \to Z$ in all possible metric spaces Z. It is known that this distance function is a metric on the family of isometry classes of non-empty compact

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metric spaces. The corresponding hyperspace is usually denoted by $\mathcal{M}$ and called *Gromov–Hausdorff space* (see [9]).

The geometry of the Gromov–Hausdorff space is rather tricky and has been intensively studied by many authors, see e.g. an overview in [9]. The technique of optimal correspondences allows to prove that the space $\mathcal{M}$ is geodesic [11, 20, 22] and to describe some local and all global isometries of $\mathcal{M}$, see [21, 24]. Since finite metric spaces form an everywhere dense subset of $\mathcal{M}$, the distances to such spaces and between such spaces play an important role in the study of the geometry of $\mathcal{M}$. Important classes of such spaces are those whose non-zero distances are the same (so-called *one-distance spaces* or *simplexes*) and the spaces whose non-zero distances take only two different values (so-called *two-distance spaces*). The authors, together with S. Iliadis and D. Grigor'ev, see [14, 19], computed the distances from a bounded metric space to a simplex, and, as a particular case, the distances between any simplex and any 2-distance space, see [26]. It turns out that the Gromov–Hausdorff distance from a metric space X to a simplex somehow "feels" a geometry of partitions of the space X. The latter explains some relations between the Gromov–Hausdorff distance and Discrete Geometry problems.

Discrete Geometry problems

In 1933 K. Borsuk posed the following question: Into how many parts must one partition an arbitrary subset of Euclidean space in order to obtain pieces with smaller diameters? He made the following famous conjecture: Any bounded, non-single-point subset of $\mathbb{R}^n$ can be partitioned into at most $n+1$ subsets, each of which has a smaller diameter than the initial subset. K. Borsuk himself proved this for $n = 2$ and for a ball in 3-dimensional space, [4, 5]. For $n = 3$ the conjecture was proved by J. Perkal (1947) and by H. G. Eggleston (1955), then for convex subsets with smooth boundaries in 1946 by H. Hadwiger [16, 17], and for central symmetric bodies by A. S. Riesling (1971). In 1993, however, the conjecture was refuted by J. Kahn and G. Kalai for the general case, see [31]. They constructed a counterexample in dimension $n = 1325$, and also proved that the conjecture does not hold for all $n > 2014$. This estimate was consistently improved by Raigorodskii, $n \geq 561$, Hinrichs and Richter, $n \geq 298$, Bondarenko, $n \geq 65$, and Jenrich, $n \geq 64$, see details in a review [37]. Note that all examples are finite subsets of the corresponding spaces, and the best known results of Bondarenko [7] and Jenrich [30] are the 2-distance subsets of the unit sphere.

On the other hand, Lusternik and Schnirelmann [34], and somewhat later independently Borsuk [4, 5], see also [42], have shown that the standard sphere and the standard ball in $\mathbb{R}^n$, $n \geq 2$, cannot be partitioned into $m \leq n$ subsets with smaller diameters. The smallest possible number of parts of smaller diameters required to partition the sphere and ball into $\mathbb{R}^n$ is therefore equal to $n + 1$.

In paper [27] we have formulated a generalized Borsuk problem that refers to an arbitrary bounded metric space X and its partitions with arbitrary cardinality m (not necessary finite), and given a criterion for solving the Borsuk problem in terms of the Gromov–Hausdorff distance. It is shown that to prove the existence of an m-partition into subsets with smaller diameter it is sufficient to calculate the Gromov–Hausdorff distance from the space X to a simplex with cardinality m and a diameter smaller than X. As a corollary, a solution of the Borsuk problem for a 2-distance space X with distances $a < b$ is obtained in terms of the clique cover number of the simple graph G with vertex set X, whose vertices x and y are connected by an edge if and only if $|xy| = a$.

Recall that a *clique cover* of a given simple graph is a cover of the graph's vertex set by subsets where two vertices are adjacent. Each such subset is called *clique* and is a vertex set of a complete subgraph, which is also called *clique*. The minimum k for which a k-elementary clique cover exists is called *clique cover number* of the given graph. Furthermore, a *graph coloring* is an assignment of labels (traditionally called "colors") to vertices of a graph in such a way that no two adjacent vertices have the same color. The smallest number of colors necessary to color a graph is called its *chromatic number*. It is not difficult to see that the clique cover can be considered as a graph coloring of the complement graph, so that the clique cover number of a graph is equal to the chromatic number of the complement graph. The computation and estimation of these numbers are very difficult combinatorial problems, see an overview in [33]. In [27] we calculated the clique cover number of a simple graph and the chromatic number of a simple graph in terms of the Gromov–Hausdorff distance from an appropriate simplex to the 2-distance spaces constructed by the graph.

Isometrical embeddings and main result

Another classical problem we deal with is the isometric embedding problem, where the question is whether a given metric space can be isometrically embedded in a given ambient space. The most studied case is when the ambient space is Euclidean. These investigations were initiated by Cayley in 1841, see [10], and later continued by Menger [35], Schönberg [38], Blumenthal [6] and other specialists. One of Schöenberg's well-known results states that a metric space (X, d) is isometrically embeddable in Hilbert space if and only if the squared distance d^2 satisfies a list of linear inequalities. These results were later extended in the context of Banach L_p- and ℓ_p-spaces. Of particular importance for our purpose is a result of Bretagnolle, Dacunha Castelle and Krivine [8], which states that (X, d) is isometrically L_p-embeddable if and only if the same holds for every finite subspace of (X, d). The same is true for ℓ_p-embeddability, see [12].

Embeddability into other metric spaces is also interesting and is intensively studied. In 1925, P. Urysohn [40] found a separable complete metric space $\mathcal{U}$ containing an arbitrary finite metric space in such "symmetric" way that any isometry between finite subsets of $\mathcal{U}$ can be extended to the isometry of the whole $\mathcal{U}$. Now $\mathcal{U}$ is known as the universal Urysohn space. In 1935, K. Kuratowski [32] constructed an isometric embedding of a metric space X into the Banach space $C_b(X)$ of bounded continuous functions on X endowed with sup-metric. In the case of separable metric spaces, the same constriction can be used to obtain an embedding into the Banach space $\ell_\infty(X)$ of all bounded sequences on X with the sup-metric. Another famous result that should be mentioned is the J. Nash Theorem [36], which guarantees the embeddability of any Riemannian manifold into a suitable Euclidean space. Recently, in collaboration with S. Iliadis, see [21, 29], the authors have constructed embeddings of an arbitrary finite metric space into the Gromov–Hausdorff space $\mathcal{M}$ and of any bounded metric space into the Gromov–Hausdorff metric class (see details in [29]).

In this paper we are interested in the case of isometrical embeddings in a rectilinear space. Based on the results just mentioned, it suffices to consider finite metric spaces (X, d), and the embeddability of a finite pseudometric space into an ℓ_1-space as into an L_1-space is equivalent to the distance d belonging to the so-called cut cone, see definition below. This important result was first found by Assouad, see [1, 2]. On the one hand, the cut cone is defined by a finite system of linear inequalities, but in contrast to the Euclidean case, not all inequalities from this system are known in general. Therefore, other approaches are of great importance. In 1998, H.-J. Bandelt, V. Chepoi and M. Laurent [3] found that the question of the embeddability of a finite space (X, d) in an m-dimensional rectilinear space $\mathbb{R}_1^m$ can be reformulated in terms of the colorability of a certain hypergraph associated with the space (X, d).

We reduce the colorability of the hypergraph to the colorability of several simple graphs and then, using our results above on the relations between the coloring number of a simple graph and the Gromov–Hausdorff distance, we reduce the question of the embeddability of a finite space (X, d) in an m-dimensional rectilinear space to the computation of the Gromov–Hausdorff distance from special 2-distant spaces associated with (X, d) to one-distant spaces, see Main Theorem.

2. Preliminaries

Let X be an arbitrary nonempty set. Recall that a function on $d\colon X \times X \to \mathbb{R}$ is called a *metric* if it is non-negative, non-degenerate, symmetric and satisfies the triangle inequality. A set with a metric is called a *metric space*. If such a function d is allowed to take infinite values, then we call d *a generalized metric*. If we omit the non-degeneracy condition, i.e. allow $d(x, y) = 0$ for some distinct x and y, then we change the term "metric" to *pseudometric*. If ρ is only non-negative and symmetric and $\rho(x, x) = 0$ for any $x \in X$, then we call such d *a distance function*, instead of metric or pseudometric. As a rule, we write $|xy|$ for $d(x, y)$ if it is not ambiguous.

In the following, all metric spaces are endowed with the corresponding metric topology. We also use the following notations. With $\#X$ we denote the cardinality of a set X. Let X be a metric space. The closure of a subset $A \subset X$ is denoted by $\bar{A}$. For any nonempty subset $A \subset X$ and a point $x \in X$ let $|xA| = |Ax| = \inf\big\{|ax| : a \in A\big\}$. Furthermore, for $r \ge 0$:

$$B_r(x) = \big\{y \in X : |xy| \le r\big\}, \quad \text{and} \quad U_r(x) = \big\{y \in X : |xy| < r\big\},$$

and

$$B_r(A) = \big\{y \in X : |Ay| \le r\big\}, \quad \text{and} \quad U_r(A) = \big\{y \in X : |Ay| < r\big\}.$$

2.1 Hausdorff distance

Recall the basic results concerning the Hausdorff distance. The details can be found in [9]. For a set X we denote by $\mathcal{P}_0(X)$ the collection of all nonempty subsets of X.

Let X be a metric space. For each $A, B \in \mathcal{P}_0(X)$ we set

$$d_H(A,B) = \inf\bigl\{r \in [0,\infty] : A \subset B_r(B) \text{ and } B_r(A) \supset B\bigr\}.$$

It is easy to see that d_H is non-negative, symmetric and $d_H(A,A) = 0$ for any nonempty $A \subset X$, so d_H is a generalized distance on the family $\mathcal{P}_0(X)$ of all nonempty subsets of a metric space X, moreover it is a generalized pseudometric on $\mathcal{P}_0(X)$, i.e. it satisfies the triangle inequality. The function d_H is called *Hausdorff distance*.

Furthermore, we use $\mathcal{H}(X) \subset \mathcal{P}_0(X)$ to denote the set of all nonempty closed subsets of a metric space X. It is known that the Hausdorff distance d_H is a metric on $\mathcal{H}(X)$.

2.2 Gromov–Hausdorff distance

Let X and Y be metric spaces. A triple (X', Y', Z) consisting of a metric space Z and its two subsets X' and Y', which are isometric to X and Y respectively, is called a *realization of the pair* (X,Y). Set

$$d_{GH}(X,Y) = \inf\bigl\{r \in \mathbb{R} : \exists \text{ realization } (X',Y',Z) \text{ of } (X,Y) \text{ with } d_H(X',Y') \le r\bigr\}.$$

The value $d_{GH}(X,Y)$ is obviously non-negative, symmetric and $d_{GH}(X,X) = 0$ for any metric space X. Thus, d_{GH} is a generalized distance function on each set of metric spaces.

Definition 2.1. The value $d_{GH}(X,Y)$ is called *the Gromov–Hausdorff distance* between the metric spaces X and Y.

It is known that the function d_{GH} is a generalized pseudometric on every set of metric spaces. If the diameters of all spaces in the family are bounded by the same number, then d_{GH} is a pseudometric. In general, d_{GH} is not a metric, it may equal zero for distinct metric spaces. However, if we restrict ourselves to compact metric spaces considered up to an isometry, then d_{GH} is a metric.

For specific calculations of the Gromov–Hausdorff distance, other equivalent definitions of this distance are useful.

Recall that *a relation* between sets X and Y is defined as a subset of the Cartesian product $X \times Y$. Similar to the mappings, for each $\sigma \in \mathcal{P}_0(X \times Y)$ and for every $x \in X$ and $y \in Y$ the *image* $\sigma(x) := \bigl\{y \in Y : (x,y) \in \sigma\bigr\}$ is defined of any $x \in X$ and the *pre-image* $\sigma^{-1}(y) = \bigl\{x \in X : (x,y) \in \sigma\bigr\}$ of any $y \in Y$. In addition, for $A \subset X$ and $B \subset Y$ their *image* and *pre-image* are defined as the union of the images and pre-images of all their elements.

A relation R between X and Y is called *correspondence* if every $x \in X$ has a non-empty image and every $y \in Y$ has a non-empty pre-image. The correspondence can therefore be regarded as a surjective multivalued mapping. With $\mathcal{R}(X,Y)$ we denote the set of all correspondences between X and Y.

If X and Y are metric spaces, then for each relation $\sigma \in \mathcal{P}_0(X \times Y)$ its *distortion* $\operatorname{dis}\sigma$ is defined as follows

$$\operatorname{dis}\sigma = \sup\Bigl\{\bigl||xx'| - |yy'|\bigr| : (x,y),\, (x',y') \in \sigma\Bigr\}.$$

The most important known result about the relation between the correspondences and the Gromov–Hausdorff distance is the following equality

$$d_{GH}(X,Y) = \frac{1}{2}\inf\bigl\{\operatorname{dis} R : R \in \mathcal{R}(X,Y)\bigr\},$$

which is valid for all metric spaces X and Y, see for example [9].

For arbitrary nonempty sets X and Y, a correspondence $R \in \mathcal{R}(X,Y)$ is called *irreducible* if it is a minimal element of the set $\mathcal{R}(X,Y)$ with respect to the order given by the inclusion relation. The set of all irreducible correspondences between X and Y is denoted by $\mathcal{R}^0(X,Y)$. It can be shown [23] that for every $R \in \mathcal{R}(X,Y)$ there exists $R^0 \in \mathcal{R}^0(X,Y)$ such that $R^0 \subset R$. In particular, $\mathcal{R}^0(X,Y) \neq \emptyset$. Therefore, for any metric spaces X and Y the following equality holds

$$d_{GH}(X,Y) = \frac{1}{2}\inf\bigl\{\operatorname{dis} R \mid R \in \mathcal{R}^0(X,Y)\bigr\}.$$

Here are some simple cases of exact calculation and estimation of the Gromov–Hausdorff distance.

Example 2.2. Let Y be an arbitrary ε-net of a metric space X. Then

$$d_{GH}(X,Y) \le d_H(X,Y) \le \varepsilon.$$

Every compact metric space is thus approximated (according to the Gromov-Hausdorff metric) with any accuracy by finite metric spaces.

With Δ_1 we denote a single-point metric space.

Example 2.3. For every metric space X the following holds $d_{GH}(\Delta_1, X) = \frac{1}{2}\operatorname{diam} X$.

Example 2.4. Let X and Y be metric spaces, and let the diameter of one of them be finite. Then $d_{GH}(X,Y) \ge \frac{1}{2}|\operatorname{diam} X - \operatorname{diam} Y|$.

Example 2.5. Let X and Y be metric spaces, then

$$d_{GH}(X,Y) \le \frac{1}{2}\max\{\operatorname{diam} X, \operatorname{diam} Y\},$$

in particular, if X and Y are bounded metric spaces, then $d_{GH}(X,Y) < \infty$.

With *simplex* we denote a metric space whose non-zero distances are all equal, i.e. a one-distance space. If m is an arbitrary cardinal number, we denote by Δ_m a simplex that contains m points and whose non-zero distances are all equal to 1. Thus, $\lambda\Delta_m$, $\lambda > 0$, is a simplex whose non-zero distances are equal to λ. Furthermore, $0\,X$ coincides with Δ_1 by definition. The following result holds, for a proof see [27]

Proposition 2.6. *Let X be an arbitrary metric space, $m > \#X$ a cardinal number, and $\lambda \ge 0$, then $2d_{GH}(\lambda\Delta_m, X) = \max\{\lambda, \operatorname{diam} X - \lambda\}$.*

The case $2 \le m \le \#X$ is rather more delicate, see details in [14, 19].

2.3 Elements of Graph Theory

A *simple graph* is a pair $G = (V, E)$ consisting of two sets V and E, which are called *vertex set* and the *edge set* of the graph G, respectively; elements of V are called *vertices* and the ones of E are called *edges* of the graph G. The set E is a subset of the family of two-element subsets of V. If V and E are finite sets, then the graph G is called *finite*.

It is convenient to use the following notations:

- If $\{v, w\} \in E$ is an edge of the graph G, then we simply write it as vw or wv; furthermore, we say that an edge vw *connects the vertices* v *and* w, and that v and w are *the vertices of the edge* vw;

- We write $V(G)$ and $E(G)$ for the vertex set and the edge set of a graph G to emphasize which graph is being considered.

Two vertices $v, w \in V(G)$ are called *adjacent* if $vw \in E(G)$. Two different edges $e_1, e_2 \in E(G)$ are called *adjacent* if they have a common vertex, i.e. if $e_1 \cap e_2 \neq \emptyset$. Each edge $vw \in E(V)$ and its vertex, i.e. v or w, are referred to as *incident* to each other. The cardinal number of edges incident to a vertex v is called *degree of vertex* v and is denoted by $\deg v$.

We also need some set-theoretical operations on graphs. They are usually defined in an intuitively clear way in terms of vertex and edge sets. For example, if $G = (V, E)$ is a graph and e is a two-element subset of V, then $G \cup e = \big(V, E \cup \{e\}\big)$; similarly, for $e \in E$ put $G \setminus e = \big(V, E \setminus \{e\}\big)$.

The concept of hypergraph naturally generalizes the concept of graph. Namely, a *hypergraph* $H = (V, E)$ is a pair consisting of a *vertex set* V and an *edge set* E, where E is a family of nonempty subsets of V. As with ordinary graphs, an element of V is called *vertex* and an element of E is called *edge*.

2.4 Generalized Borsuk problem

The classical Borsuk problem deals with the partitions of subsets of Euclidean space into parts with smaller diameters. We generalize the Borsuk problem to arbitrary bounded metric spaces and partitions of arbitrary cardinality. Let X be a bounded metric space, m a cardinal number such that $2 \leq m \leq \#X$, and $D = \{X_i\}_{i \in I}$ a partition of X into m nonempty subsets. We say that D is a partition of X into subsets with *strictly smaller diameters* if there exists $\varepsilon > 0$ such that $\operatorname{diam} X_i \leq \operatorname{diam} X - \varepsilon$ for all $i \in I$.

The *Generalized Borsuk problem*: Is it possible to partition a bounded metric space X into a given, probably infinite, number of subsets, each of which has a diameter strictly smaller than X?

In [27] a solution to this problem is given in the terms of the Gromov–Hausdorff distance.

THEOREM 2.7. *Let X be an arbitrary bounded metric space and m a cardinal number such that $2 \leq m \leq \#X$. Choose an arbitrary number $0 < \lambda < \operatorname{diam} X$, then X can be partitioned into m subsets with strictly smaller diameters if and only if $2d_{GH}(\lambda\Delta_m, X) < \operatorname{diam} X$. If not, then $2d_{GH}(\lambda\Delta_m, X) = \operatorname{diam} X$.*

2.5 Clique cover number and chromatic number of a simple graph

Recall that a subgraph of an arbitrary simple graph G is called *a clique* if any its two vertices are connected by an edge, i.e. the clique is a subgraph which is itself a complete graph. Note that each single-vertex subgraph is a single-vertex clique. For simplicity, the vertex set of a clique is also referred to as *clique*.

On the set of all cliques an ordering with respect to inclusion is naturally defined, and therefore a family of maximal cliques is uniquely defined based on the above; this family forms *a covering of the graph* G in the following sense: the union of all vertex sets of all maximal cliques coincides with the vertex set $V(G)$ of the graph G.

If we do not restrict ourselves to maximal cliques, we can generally find other families of cliques covering the graph G. One of the classical problems in graph theory is to calculate the minimal possible number of cliques covering a given finite simple graph G. This number is called *clique cover number* and is often denoted by $\theta(G)$. It is easy to see that the value $\theta(G)$ also corresponds to the smallest number of cliques whose vertex sets form a partition of $V(G)$.

Another popular problem is to find the smallest possible number of colors necessary to color the vertices of a simple finite (hyper-) graph G without monochromatic edges. This number is denoted by $\gamma(G)$ and is called *chromatic number of the (hyper-) graph G*.

For a simple graph G, we denote by G' its *complement graph*, i.e. the graph with the same vertex set and the complementary set of edges (two vertices of G' are adjacent if and only if they are not adjacent in G). It is easy to see that $\theta(G) = \gamma(G')$ for any simple finite graph G, see for example [41].

Let $G = (V, E)$ be an arbitrary finite graph. Let there be two positive real numbers $a < b \le 2a$ and define a metric on V as follows: the distance between adjacent vertices is equal to a, and the distance between nonadjacent vertices is equal to b. Then a subset $V' \subset V$ has diameter a if and only if $G(V') \subset G$ is a clique. This implies that the clique cover number of G equals the smallest possible cardinality of partitions of the metric space V into subsets with (strictly) smaller diameter. However, this number was calculated in Theorem 2.7. We therefore obtain the following result.

COROLLARY 2.8. *Let $G = (V, E)$ be an arbitrary finite graph. Fix two positive real numbers $a < b \le 2a$ and define a metric on V as follows: the distance between adjacent vertices equals a, and the distance between nonadjacent ones equals b. Let m be the greatest positive integer k such that $2d_{GH}(a\Delta_k, V) = b$ (in the case when there is no such k, we put $m = 0$). Then $\theta(G) = m + 1$.*

Because of the duality between clique cover and chromatic numbers, we get the following dual result.

COROLLARY 2.9. *Let $G = (V, E)$ be an arbitrary finite graph. Fix two positive real numbers $a < b \le 2a$ and define a metric on V as follows: the distance between adjacent vertices equals b, and the distance between nonadjacent ones equals a. Let m be the greatest positive integer k such that $2d_{GH}(a\Delta_k, V) = b$ (in the case when there is no such k, we put $m = 0$). Then $\gamma(G) = m + 1$.*

Corollary 2.10. *Let $G = (V, E)$ be an arbitrary finite graph. Fix two positive real numbers $a < b \leq 2a$ and define a metric on V as follows: the distance between adjacent vertices equals b, and the distance between nonadjacent ones equals a. If $2d_{GH}(a\Delta_k, V) = b$, then $\gamma(G) > k$.*

2.6 Embeddings into rectilinear spaces

We recall that the rectilinear m-dimensional space $\mathbb{R}^m_1$ is the real m-dimensional space $\mathbb{R}^m$ endowed with the distance function generated by the norm $\|y\|_1 = \sum_i |y_i|$ for $y = (y_1, \dots, y_m) \in \mathbb{R}^m$. A pseudometric space (X, d) is called *m-embeddable in a rectilinear space* if there exists an isometrical embedding $f\colon X \to \mathbb{R}^m_1$. The latter means that $d(a, b) = \big\|f(a) - f(b)\big\|_1$ for all $a, b \in X$. As we mentioned above, the m-embeddability problem of an arbitrary pseudometric space is equivalent to that for its finite subspaces (see a proof in [12]).

To describe finite pseudometric spaces embeddable into rectilinear spaces, we need to recall the concepts of cut pseudometrics and cut cones. Let X be a finite set consisting of n elements. Without loss of generality, we set $X = \{1, \dots, n\}$. For a proper subset $S \subset X$, the pair $\{S, S'\}$, where S' stands for $X \setminus S$, is called a *cut*. For a cut $c = \{S, S'\}$, define a pseudometric δ_c on X as follows: $\delta_c(i, j) = 1$ if $\#\big(S \cap \{i, j\}\big) = 1$, and $\delta_c(i, j) = 0$ otherwise. The pseudometric δ_c is referred to as the *cut metric*, which corresponds to the cut $c = \{S, S'\}$.

Recall that every pseudometric d on $X = \{1, \dots, n\}$ defines the square symmetric matrix $\big(d(i, j)\big)$ with $d(i, i) = 0$, which can be given by the vector $\big(d(1, 2), \dots, d(n - 1, n)\big)$ in $\mathbb{R}^N$, $N = n(n - 1)/2$. The set of all such vectors is a convex cone in $\mathbb{R}^N$ called the *metric cone*. Let $\mathcal{C}$ be a family of cuts of X, and $\lambda\colon \mathcal{C} \to \mathbb{R}$ a mapping such that $\lambda_c := \lambda(c) > 0$ for all $c \in \mathcal{C}$. The pseudometric $d(\mathcal{C}, \lambda) = \sum_{c\in\mathcal{C}} \lambda_c \delta_c$ is called a *cut metric*, corresponding to the family $\mathcal{C}$ of cuts. Consider all such metrics $d(\mathcal{C}, \lambda)$ over all families of $\mathcal{C}$ and all mappings λ. They form another cone in $\mathbb{R}^N$, which is called *cut cone* and is denoted by CUT_n. The following criterion is obtained in [1].

Assertion 2.11. *A finite metric space (X, d) is isometrically embeddable in a rectilinear space if and only if $d \in \mathrm{CUT}_{\#X}$.*

Remark 2.12. The same criterion works for L_1-embeddability of finite metric spaces, i.e., a finite metric space (X, d) is isometrically embeddable in an L_1-space if and only if $d \in \mathrm{CUT}_{\#X}$ (and if and only if it is embeddable in a rectilinear space).

Remark 2.13. The condition of Assertion 2.11 seems simple, but it is rather difficult to verify. Moreover, it is proved that the problem is NP-complete.

If the answer is positive, that is, if a metric space (X, d) turns out to be embeddable, then the question on the minimal admissible dimension of the rectilinear space arises naturally. This minimal dimension is referred as *ℓ_1-dimension* of the space (X, d).

To state the coloring criterion found in [3] we need to construct a special hypergraph associated with a cut system $\mathcal{C}$ of a set X. Two cuts $\{A, A'\}$ and $\{B, B'\}$ are said to be *incompatible* if all four intersections $A \cap B$, $A \cap B'$, $A' \cap B$, and $A' \cap B'$ are non-empty. Three cuts $\{A, A'\}$, $\{B, B'\}$, and $\{C, C'\}$ are said to form an *asteroid triplet* if one can choose one set $\tilde{A} \in \{A, A'\}$, $\tilde{B} \in \{B, B'\}$, and $\tilde{C} \in \{C, C'\}$ from each cut in such a way that the three resulting sets $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are pairwise disjoint. The hypergraph with vertex set $\mathcal{C}$ whose edges are all incompatible pairs and all asteroid triplets is called *nesting hypergraph of* $\mathcal{C}$ and is denoted by $\Gamma(\mathcal{C})$.

One says that a hypergraph $G = (V, E)$ is *m-colorable* if there exists a coloring with m colors without monochromatic edges. In [3] the following result is proved.

Assertion 2.14. *Let (X, d) be a finite metric space, and*

$$d = \sum_{c \in \mathcal{C}} \lambda_c \delta_c, \qquad \lambda_c > 0,$$

where $\mathcal{C}$ is a cut family on X. Then (X, d) is embeddable in m-dimensional rectilinear space if and only if the corresponding nesting hypergraph $\Gamma(\mathcal{C})$ is m-colorable. In particular, the chromatic number of $\Gamma(\mathcal{C})$ equals the ℓ_1-dimension of (X, d).

3. Embeddings an Gromov–Hausdorff distance

The main result of the paper is based on two findings: Assertion 2.14 and Corollary 2.9.

Let $\mathcal{C}$ be an arbitrary cut family of a finite set X and $\Gamma(\mathcal{C})$ the corresponding nesting hypergraph. Note that for each of its edges $\{a, b, c\}$ corresponding to an asteroid triplet, none of the pairs $\{a, b\}$, $\{b, c\}$ and $\{a, c\}$ forms an edge of $\Gamma(\mathcal{C})$. Construct a family of simple graphs with vertex set $\mathcal{C}$ as follows: For each asteroid triplet $\{a, b, c\}$, choose one of the pairs $\{a, b\}$, $\{b, c\}$, $\{a, c\}$, add it to the edge set and delete the triplet. Note that such a pair could belong to multiple asteroid triplets, but if it is chosen multiple times, we add it to the edge set only once to avoid multiple edges. As a result, we get at most 3^k simple graphs $\{G_i\}$, where k stands for the number of asteroid triplets in the nesting hypergraph. We set $\mathcal{G}(\mathcal{C}) = \{G_i\}$.

Remark 3.1. If the nesting hypergraph does not contain an asteroid triplet, then it is itself a simple graph and the family $\mathcal{G}(\mathcal{C})$ consists of a single element $\Gamma(\mathcal{C})$.

Lemma 3.2. *Let $\mathcal{C}$ be an arbitrary cut family of a finite set X and $\Gamma(\mathcal{C})$ a corresponding nesting hypergraph, and let $\mathcal{G}(\mathcal{C}) = \{G_i\}$ be the family of simple graphs constructed above. Then $\Gamma(\mathcal{C})$ is m-colorable if and only if the family $\mathcal{G}(\mathcal{C})$ contains an m-colorable simple graph.*

Proof. Let $\Gamma(\mathcal{C})$ be m-colorable, and fix some its m-coloring ξ without monochromatic edges. Then each asteroid triplet contains at least two differently colored vertices. Construct a simple graph G by deleting each asteroid triplet and adding an edge connecting these two vertices. If the vertices of such an edge belong to multiple asteroid triplets, then we add this edge at most once. The resulting simple graph belongs to $\mathcal{G}(\mathcal{C})$ and ξ is its coloring without monochromatic edges.

Conversely, let $G \in \mathcal{G}(\mathcal{C})$ and ξ be its coloring without monochromatic edges. Then ξ is also a coloring of the nesting hypergraph $\Gamma(\mathcal{C})$ that contains no monochromatic edges. In fact, each asteroid triplet of $\Gamma(\mathcal{C})$ contains an edge of G and thus at least two differently colored vertices. □

COROLLARY 3.3. *Let (X,d) be a finite metric space, and $d = \sum_{c\in\mathcal{C}} \lambda_c \delta_c$, $\lambda_c > 0$, where $\mathcal{C}$ is a cut family on X. Then (X,d) is embeddable in an m-dimensional rectilinear space if and only if the corresponding family $\mathcal{G}(\mathcal{C})$ of simple graphs contains an m-colorable graph. The ℓ_1-dimension of (X,d) is equal to $\min_G \gamma(G)$, where the minimum over all $G \in \mathcal{G}(\mathcal{C})$ is taken.*

We now fix two positive real numbers $0 < a < b \le 2a$, and construct for each graph G from $\mathcal{G}(\mathcal{C})$ a 2-distant metric d_G on $\mathcal{C}$ as follows: We set $d_G(s,t) = b$ if and only if s and t are adjacent, and $d_G(s,t) = a$ otherwise. With $\mathcal{C}_G$ we denote the resulting metric space $(\mathcal{C}, d_G)$.

THEOREM 3.4. *Let (X,d) be a finite metric space, and $d = \sum_{c\in\mathcal{C}} \lambda_c \delta_c$, $\lambda_c > 0$, where $\mathcal{C}$ is a cut family on X. Then the ℓ_1-dimension of (X,d) is equal to m if and only if m is the smallest positive integer for which $\min_G 2d_{GH}(a\Delta_m, \mathcal{C}_G) < b$, where minimum is taken over the family $\mathcal{G}(\mathcal{C})$, and $a\Delta_m$ stands for the one-distance metric space with nonzero distance a.*

Proof. It is easy to see that $2d_{GH}(a\Delta_m, \mathcal{C}_G) \le b$ for all positive integers m and every $G \in \mathcal{G}(\mathcal{C})$. And if $m \ge \#\mathcal{C}$, then for every $G \in \mathcal{G}(\mathcal{C})$ the distortion of any correspondence between $a\Delta_m$ and $\mathcal{C}_G$ such that the preimages of different cuts do not intersect is smaller than b. Therefore, the set of positive integers is such that $\min_G 2d_{GH}(a\Delta_m, \mathcal{C}_G) < b$ is not empty.

Let m be the smallest positive integer for which $\min_G 2d_{GH}(a\Delta_m, \mathcal{C}_G) < b$, $G \in \mathcal{G}(\mathcal{C})$. Let us start with the case $m = 1$. Recall that $2d_{GH}(a\Delta_1, \mathcal{C}_G) = \operatorname{diam} \mathcal{C}_G$, see Example 2.3, therefore in this case all distances in the 2-distance space $\mathcal{C}_G$ are equal to a, and therefore the graph G is empty (i.e. it has no edges). The nesting graph $\Gamma(\mathcal{C})$ is therefore also empty and can be colored in one color. Therefore, (X,d) is embeddable in the straight line $\mathbb{R}^1 = \mathbb{R}^1_1$ and therefore the ℓ_1-dimension of (X,d) is equal to 1.

Now let $m \ge 2$. Due to the assumptions, there is $G_0 \in \mathcal{G}(\mathcal{C})$ such that, for all k, $1 \le k < m$, and all $G \in \mathcal{G}(\mathcal{C})$: $2d_{GH}(a\Delta_m, \mathcal{C}_{G_0}) < b$, and $2d_{GH}(a\Delta_k, \mathcal{C}_G) = b$. This means that $(m-1)$ is the largest positive integer for which $2d_{GH}(a\Delta_k, \mathcal{C}_{G_0}) = b$, and therefore, due to Corollary 2.9, $\gamma(G_0) = m$, and (X,d) is embeddable in an m-dimensional rectilinear space according to Corollary 3.3.

Furthermore, if (X,d) is embeddable in a k-dimensional rectilinear space, then due to Corollary 3.3 the family $\mathcal{G}(\mathcal{C})$ contains a k-colorable graph G. Consider the corresponding 2-distant metric space $\mathcal{C}_G = (\mathcal{C}, d_G)$. According to Corollary 2.10, $2d_{GH}(a\Delta_k, \mathcal{C}_G) < b$, and therefore $\min_G 2d_{GH}(a\Delta_k, \mathcal{C}_G) < b$. So $k \ge m$, due to the assumptions, and therefore m is the ℓ_1-dimension of (X,d).

Conversely, let ℓ_1-dimension of (X,d) be equal to m. As already shown above, the latter implies that $\min_G 2d_{GH}(a\Delta_m, \mathcal{C}_G) < b$. If there exists $k < m$ such that

$\min_G 2d_{GH}(a\Delta_m, \mathcal{C}_G) < b$, then by virtue of the direct statement (X, d) is embeddable in a k-dimensional rectilinear space, and the ℓ_1-dimension of (X, d) is less than m, a contradiction. The theorem is proven. □

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