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GREEN FUNCTIONS FOR VARIOUS BLACK HOLE METRICS

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Abstract. A few models in general relativity concerning to black holes are considered. We studied the Green function for locally close points in the Schwarzschild (symmetric non-rotating and uncharged black hole), Reissner-Nordström (charged non-rotating black hole), Schwarzschild-de Sitter (black hole with a positive cosmological constant), Reissner-Nordström-de Sitter (when a constant electric charge is added to the cosmological term), Hayward (spherically symmetric non-rotating uncharged black hole having no singularity), Bardeen (for spherically symmetric black hole being a source of electric field which does not have a singularity but have the event horizon) metrics. The consideration is based on the Hadamard-WKB method. The Padé approximation is used for the Green function construction.

1. Introduction

The properties of the space-time and particles in the vicinity of black holes have been of particular interest since the beginning of the general theory of relativity. There are many intriguing questions that are still open today, e.g. gravitational lensing near a black hole [10], influence of dark energy on geodesics [11], problems related to modes, horizon, black hole accretion disk theory [2, 4, 18, 22]. Many of these problems are related to the determination of the Green's function for various metrics [5, 19]. The main benefit of the Green's function for the problem in question is that it allows to describe the motion of a particle (in the model - point particle) in a curved spacetime [20]. This problem is basic for many gravitational problems.

The explicit form of the energy-momentum tensor for a black hole can be determined using the Green's function of the wave operator:

$$D = (\Box - m_{field}^2) + P,$$

where $\Box$ is the d'Alembert operator (wave operator), P is the potential function and m_{field} is the field mass. There are several methods to obtain the Green's function

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in this case, but we will only consider the Hadamard-WKB method. This method was first applied by Anderson and Hu [1] for the Schwarzschild metric and was subsequently further developed and applied in the study of the Narya metric [6].

In this paper, the Green's functions for the Schwarzschild, Reissner-Nordström, Schwarzschild-de Sitter, Reissner-Nordström-de Sitter, Hayward and Bardeen metrics corresponding to the different types of black holes. The algorithm for determining the residual part of the Green's function is based on [13]. Wolfram Mathematica 11.3 was used as the computational environment. Finally, results for these metrics were obtained using the Pad'e approximation of Green's functions.

2. Problem statement

Let us formulate the problem of determining the Green's function using the Hadamard-WKB method. For a curved space, the fundamental object is the retarded Green's function, where x' is a space-time point that lies in the past with respect to the point x. The delayed Green's function is a solution of the wave equation [12, 17]:

$$DG_{ret}(x, x') = -4\pi\delta(x, x').$$

If we assume that the points and are locally close, then we can represent this solution in Hadamard form in four-dimensional space-time:

$$G_{ret}(x, x') = \theta_{\text{-}}(x, x') \left\{ U(x, x')\delta(\sigma(x, x')) - V(x, x')\theta(-\sigma(x, x')) \right\},$$

where $\theta_{\text{-}}(x, x')$ is the Heaviside step function, $\delta(x, x')$ is the Dirac delta function, $U(x, x')$ is responsible for the singular part of the Green function, $V(x, x')$ — for the residual part, $\sigma(x, x')$ is the Synge's world function, which is equal to half the length of the geodesic between the points x and x' [23].

We are interested in the function $V(x, x')$, which is the residual part of Green's function and reflects the interactions of the field with the space-time geometry.

3. Problem solution

Let us expand the function $V(x, x')$ into a power series:

$$V(x, x') = \sum_{i,j,k=0}^{\infty} v_{ijk}(r)(t - t')^{2i}(\cos\gamma - 1)^j (r - r')^k,$$

where γ is the angle between x and x'.

We need to find the coefficients $v_{i,j,k}(r)$. Anderson and Hu have already calculated them for the Schwarzschild metric [1]. However, by applying the Hadamard-WKB method, the entire family of the following metrics can be considered:

$$ds^2 = -f(r)dt^2 + (f(r))^{-1}dr^2 + g(r)d\Omega_2^2,$$
$$d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2, \quad g(r) = r^2.$$

In this paper, we consider metrics with different $f(r)$ but the same $g(r)$.

First, we express the resulting Green's function in the Hadamard approximation as the real part of the Euclidean Green's function $G_E(t, x, t', x')$:

$$\mathrm{Re}\left[G_E(t, x, t', x')\right] = \frac{1}{2\pi}\left(\frac{U(x, x')}{\sigma(x, x')}\right) + V(x, x')\ln\left(|\sigma(x, x')|\right) + W(x, x').$$

In the case that the points x and x' are further apart in time than in space, we can express the Synge function $\sigma(x, x') = -\frac{1}{2}f(r)(t-t')^2 + O\left[(x-x')^3\right]$. Then the logarithmic part of the Euclidean Green's function has the form: $\frac{1}{\pi}V(x, x')\ln(t-t')$.

To find the residual part $V(x, x')$ of the Green's function, it is therefore sufficient to determine the coefficient of the logarithmic part of the Euclidean Green's function. For spaces with spherical symmetry, the Euclidean Green's function has the following integral representation

$$G_E(t, x, t', x') = \frac{1}{\pi}\int\limits_0^\infty d\omega \cos\left[\omega(t-t')\right]\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)C_{\omega l}p_{\omega l}(r_<)q_{\omega l}(r_>). \quad (1)$$

Here γ is the angle between x and x', $P_l((\cos\gamma))$ is the Legendre polynomial of order l, $C_{\omega l}$ is the normalization constants, $r_>$ and $r_<$ denote greater (resp. smaller) values of r and r', $p_{\omega l}$ and $q_{\omega l}$ (depending on ω) are solutions of the homogeneous scalar wave equation:

$$f\frac{d^2S}{dr^2} + \frac{1}{g}\frac{d}{dr}(fg)\frac{dS}{dr} - \left(\frac{\omega^2}{f} + \frac{l(l+1)}{g} + m_{field}^2 + \xi R\right)S = 0, \quad (2)$$

where m_{field} is the field mass, ξ is the coupling constant ($\xi = \frac{d-2}{4(d-1)}$, where d is the space dimension, i.e. $\xi = \frac{1}{6}$), R is the Ricci scalar [14].

Taking advantage of the fact that:

$$\int\limits_\lambda^\infty d\omega\cos\left[\omega(t-t')\right]\frac{1}{\omega^{2n+1}} = \frac{-1}{(2n)!}(t-t')^{2n}\log(t-t') + \ldots, \quad (3)$$

we can ignore all smooth terms and obtain the following form for the cos-Fourier transform $V(x, x', \omega)$ of the function $V(x, x')$ (considering (3) we only need one term of the expansion of $V(x, x', \omega)$ in powers of ω):

$$V(x, x', \omega) = \sum_{l=0}^{\infty}(2l+1)P_l(\cos(\gamma))C_{\omega l}p_{\omega l}(r_<)q_{\omega l}(r_>),$$

where $P_l(\cos(\gamma))$ is the Legendre polynomial.

Next, we look for $B(r, r') = C_{\omega l}p_{\omega l}(r')q_{\omega l}(r)$. We define it using a power series:

$$B(r, r') = \beta(r) + \alpha(r)(r'-r) + \left\{\left[\frac{2(\eta(r)+\chi^2(r))}{(fg)^2}\right]\beta(r) - \left[\ln(fg)\right]\alpha(r)\right\}\frac{(r'-r)}{2} + \ldots,$$

where

$$\beta(r) = C_{\omega l}p_{\omega l}(r)q_{\omega l}(r), \qquad \alpha(r) = \frac{\beta'(r)}{2} + \frac{1}{2f(r)g(r)},$$

$$\chi^2(r) = \omega^2 g^2 + fg\left(l+\frac{1}{2}\right)^2, \qquad \eta(r) = -\frac{1}{4}fg + (m_{field}^2 + \xi R)fg^2. \tag{4}$$

Using the equations (3) and (4), we conclude that $\beta(r)$ satisfies the following differential equation:

$$fg\frac{d}{dr}\left(fg\frac{d\sqrt{\beta}}{dr}\right) - (\eta+\chi^2)\sqrt{\beta} + \frac{1}{4\beta^{3/2}} = 0. \tag{5}$$

Equation (5) can be converted into the following form suitable for iterations:

$$\beta = \frac{1}{2\chi}\left(1 - \frac{1}{\chi^2}\left(\frac{1}{\sqrt{\beta}}fg\frac{d}{dr}\left(fg\frac{d\sqrt{\beta}}{dr}\right) - \eta(r)\right)\right)^{-\frac{1}{2}}. \tag{6}$$

Since the quasi-locality of the Green's function is determined by large values of ω, l, we represent $\beta(r)$ as a series in $\varepsilon = \frac{1}{\chi}$: $\beta = \varepsilon\beta_0 + \varepsilon^2\beta_1 + \ldots, \qquad \beta_0 = \frac{1}{2}$.

If you insert this series into (6) and use the method of iterations, you get the expressions for $\beta_n(r)$:

$$\beta_n(r) = \sum_{m=0}^{2n}\frac{A_{n,m}(r)\omega^{2m}}{\chi^{2n+2m+1}}, \quad \beta = \frac{A_{0,0}}{\chi} + \frac{1}{\chi^7}\left(A_{1,0}\chi^4 + A_{1,1}\chi^2 + A_{1,2}\right) + \ldots.$$

The coefficients $A_{n,m}(r)$ result from the recursive relations for $\beta_n(r)$. Next, we find the derivative of $\beta(r)$: $\beta' = \varepsilon\beta_0' + \varepsilon^2\beta_1' + \ldots$. Using the previous relations we get

$$\beta_n'(r) = \sum_{m=0}^{2n}\left(\frac{A_{n,m}'(r)\omega^{2m}}{\chi^{2n+2m+1}} - \left(n+m+\frac{1}{2}\right)\frac{2\chi\chi' A_{n,m}(r)}{\chi^{2n+2m+3}}\right),$$

$$\beta_n'(r) = \sum_{m=0}^{2n+1} D_{n,m}\frac{\omega^{2m}}{\chi^{2n+2m+1}},$$

where

$$\begin{aligned} D_{n,m} = A_{n,m}'(r) - \left(n+m+\frac{1}{2}\right)A_{n,m}(r)\frac{(fg)'}{fg} \\ - \left(n+m-\frac{1}{2}\right)A_{n,m-1}(r)\left[(g^2)' - \frac{(fg)'}{fg}g^2\right]. \end{aligned} \tag{7}$$

Then the sum of the function series for $V(x,x')$ reduces to the following form:

$$V(x,x',\omega) = \sum_{n=0}^{\infty}\sum_{m=0}^{2n+1}\sum_{l=0}^{\infty} 2\left(l+\frac{1}{2}\right)P_l(\cos(\gamma))\frac{D_{n,m}(r)\omega^{2m}}{\chi^{2n+2m+1}}.$$

We can also simplify the Legendre polynomial as γ tending to zero:

$$V(x,x',\omega) = 2\sum_{n=0}^{\infty}\sum_{m=0}^{2n+1} D_{n,m}(r)\sum_{l=0}^{\infty}\frac{\left(l+\frac{1}{2}\right)^{2p+1}\omega^{2m}}{\chi^{2n+2m+1}}.$$

It can be seen that the Sommerfield-Watson formula can be used to convert the sum in l into two integrals:

$$\sum_{l=0}^{\infty} F\left(l+\frac{1}{2}\right) = \int_0^{\infty} F(\lambda)d\lambda - \mathrm{Re}\left(i\int_0^{\infty} \frac{2}{1+e^{2\pi\lambda}}F(i\lambda)d\lambda\right). \tag{8}$$

In our case, the first integral in (8) is real and the second integral reduces to a contour integral in the complex plane. We transform these integrals into a series in inverse powers of ω. Taking into account the integral representation (1) and the formula (3), we then arrive at the residual part of the Green's function.

In order to find a rational approximation outside the convergence domain of the power series, the Padé approximation is used (see, e.g., [8]). We follow the idea of this work.

Summarizing the results, we arrive at the following theorem.

THEOREM 3.1. *If the metric of the space has the following form*

$$ds^2 = -f(r)dt^2 + (f(r))^{-1}dr^2 + g(r)(d\theta^2 + \sin^2\theta d\phi^2), \quad g(r) = r^2,$$

then the residual part of the Green function $V(x,x')$ is the cos-Fourier transform of the term with the power $\frac{1}{\omega^{2n+1}}$ of the expansion ω of the function

$$V(x,x',\omega) = \sum_{n=0}^{\infty}\sum_{m=0}^{2n+1}\sum_{l=0}^{\infty} 2\left(l+\frac{1}{2}\right)P_l(\cos(\gamma))\frac{D_{n,m}(r)\omega^{2m}}{\chi^{2n+2m+1}},$$

where $P_l(\cos(\gamma))$ is the Legendre polynomial, γ is the angle between x and x',

$$\chi^2(r) = \omega^2 g^2 + fg\left(l+\frac{1}{2}\right)^2,$$

$$D_{n,m} = A'_{n,m}(r) - \left(n+m+\frac{1}{2}\right)A_{n,m}(r)\frac{(fg)'}{fg} - \left(n+m-\frac{1}{2}\right)A_{n,m-1}(r)\left[(g^2)' - \frac{(fg)'}{fg}g^2\right],$$

$A_{n,m}(r)$ is the coefficient of the series $\beta(r) = \sum_{n=0}^{\infty}\sum_{m=0}^{2n} \frac{A_{n,m}(r)\omega^{2m}}{\chi^{2n+2m+1}}$, $\beta(r)$ in a regular solution of the equation

$$fr^2\frac{d}{dr}\left(fr^2\frac{d\sqrt{\beta}}{dr}\right) - \left(-\frac{1}{4}fr^2 + \left(m^2_{field} + \frac{R}{6}\right)fr^4 + \chi^2\right)\sqrt{\beta} + \frac{1}{4\beta^{3/2}} = 0,$$

R is the Ricci scalar.

REMARK 3.2. In this formulation, we have taken the formula (3) into account.

4. Results and discussion

4.1 Green's function for the Schwarzschild metric

Based on the formulas obtained above, one can find the functions $V(x,x')$ for various metrics. A change in $f(r)$ leads to a change in the equation (2) and gives us $p_{\omega l}$ and $q_{\omega l}$ (depending on ω). The equation (2) transforms to more convenient form (6). Accordingly, the change in the metric leads to a change in the equation (5) and the next equation (6). The solution of (6) is sought in the form of series using the same procedure for all metrics (with the corresponding substitution of functions). This procedure is described in the previous section.

Since only a limited number of terms of the power series are used for the calculations, there may be deviations from the real data, so the Padé approximation can be used to bring the results closer to the real data. For simplicity and better visibility, we consider points where all coordinates, except time, coincide.

Let us first consider the Schwarzschild metric [1]. This metric is obtained from the simplest solution of the Einstein equations without a cosmological constant, which describes the field of a symmetric, non-rotating and uncharged black hole. The form of the function $f(r)$ in the metric tensor is $f(r) = 1 - \frac{2M}{r}$.

Let's calculate $V(t,t')$ for $M = 1$, $r = 1$. We obtain:

$$V(t,t') = 12.291dt^{30}+7.398dt^{28}+4.45dt^{26}+2.673dt^{24}+1.603dt^{22}+0.959dt^{20}+0.572dt^{18} +0.339dt^{16}+0.199dt^{14}+0.116dt^{12}+0.066dt^{10}+0.036dt^{8}+0.018dt^{6}+0.007dt^{4}.$$

Figure 1 shows the Padé approximation for given values of the parameters.

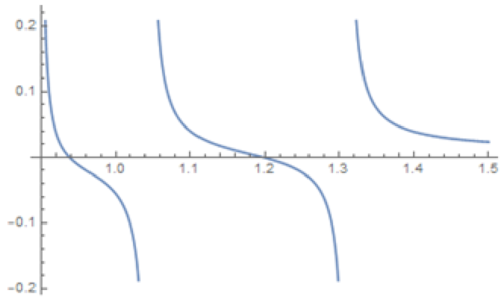

Figure 1: Padé approximation plot for $V(t,t')$ at $M = 1$, $r = 1$.

4.2 Green's function for the Reissner-Nordström metric

The formulas obtained can be used for the Reissner-Nordström metric [8]. It is similar to the Schwarzschild metric in many respects, but has one difference - it considers a charged, non-rotating black hole. The function in the metric tensor has the following form $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$.

If we set the parameters $M = 1$, $r = 1$, $Q = 1$, we obtain the following expression:

$$V(t,t') = -8.68 \times 10^{27}dt^{18} - 6.706 \times 10^{24}dt^{16} - 5.174 \times 10^{21}dt^{14} - 3.981 \times 10^{18}dt^{12} - 3.047 \times 10^{15}dt^{10} - 2.31 \times 10^{12}dt^{8} - 1.716 \times 10^{9}dt^{6} - 1.212 \times 10^{6}dt^{4} - 0.769 \times 10^{3}dt^{2}.$$

Then the Padé approximation for the given values of the parameters has the form presented in Figure 2.

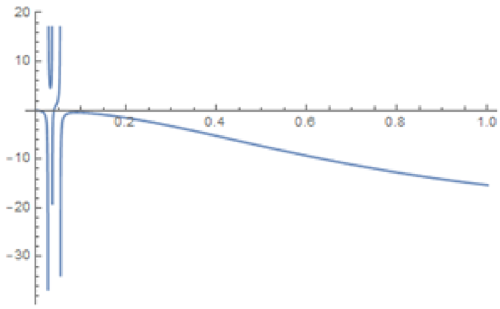

Figure 2: Padé approximation plot for $V(t,t')$ at $M=1$, $r=1$, $Q=1$.

4.3 Green's function for the Schwarzschild-de Sitter metric

The next example of an interesting metric is the Schwarzschild-de Sitter metric. Unlike the previous metrics, it is a solution to the Einstein equation for a black hole with a positive cosmological constant. This metric is the simplest solution for the case where a black hole has both an event horizon and a cosmological horizon [7]. The type of the function $f(r)$ in the metric tensor is $f(r) = 1 - \frac{2M}{r} - \frac{Lr^2}{3}$.

If you take $M=1$, $r=1$, $L=1$, you get:

$$V(t,t') = 2.468dt^{16} + 1.16dt^{14} + 0.539dt^{12} + 0.246dt^{10} + 0.109dt^{8} + 0.044dt^{6} + 0.014dt^{4}.$$

The corresponding Padé approximation is shown in Figure 3.

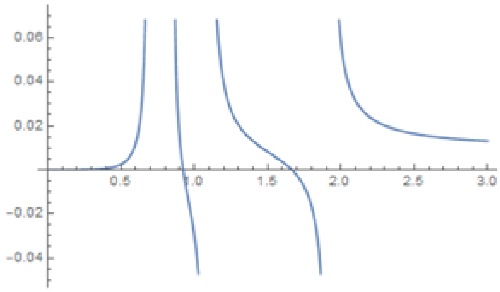

Figure 3: Padé approximation plot for $V(t,t')$ with $M=1$, $r=1$, $L=1$.

4.4 Green's function for the Reissner-Nordström-de Sitter metric

Let us consider Green's function for the case where a constant electric charge is added to the cosmological term. For this case we have the Reissner-Nordström-de Sitter metric. In this case, the function $f(r)$ is $f(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} + \frac{Q^2}{r^2}$.

The function $V(t,t')$ has the following form:

$$V(t,t') = 3.585 \times 10^{-10}dt^{10} + 1.293 \times 10^{-8}dt^{8} + 0.659 \times 10^{-7}dt^{6} + 1.805 \times 10^{-6}dt^{4} + 3.482 \times 10^{-5}dt^{2}.$$

The Padé approximation for the function $V(t,t')$ is shown in Figure 4.

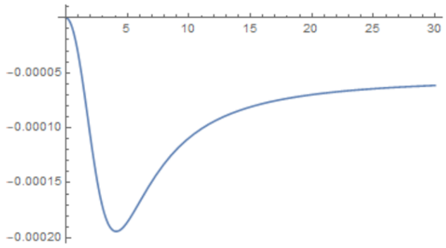

Figure 4: Padé approximation plot of $V(t,t')$ for the Reissner-Nordström-de Sitter metric with $M = 1$, $r = 10$, $Q = 1$, $\Lambda = 1$.

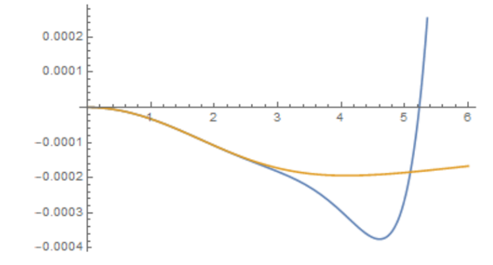

Figure 5: Plot of the $V(t,t')$ function (blue line) and Padé approximation (orange line) for the metric Reissner-Nordström-de Sitter at $M = 1$, $r = 10$, $Q = 1$, $\Lambda = 1$.

We can see that minimum points occur for both functions. This is consistent with the fact that $V(x,x')$ must be smooth and continuous. The singular points correspond to the eigenvalues of the spectrum of the operator under study. They may well correspond to the "potential wells" of the field combined with the curvature of the space.

4.5 Green's function for the Hayward metric

For the Hayward metric one has [9,15,16] $f(r) = 1 - \frac{2Mr^2}{r^3+2l^2M}$.

Here M is the mass parameter and l is the length scale parameter. Obviously $f(r)$ has no singularity. Using the Hadamard-WKB algorithm, we find the residual part of Green's function for the Hayward metric (for a spherically symmetric, non-rotating, uncharged black hole without singularity) for the following parameter values $M = 1$, $r = 10$, $l = 1$.

$$V(t,t') = 2.493 \times 10^{-8} dt^{12} + 2.282 \times 10^{-7} dt^{10} + 2.517 \times 10^{-6} dt^{8} + 5.696 \times 10^{-5} dt^{6} + 8.592 \times 10^{-5} dt^{4} + 1.77 \times 10^{-2} dt^{2} - 0.333.$$

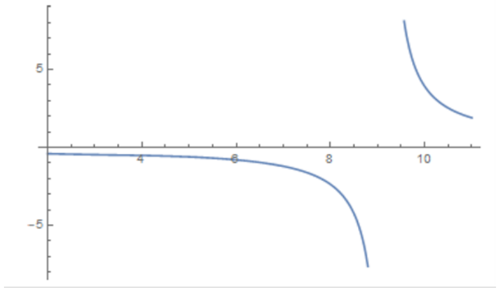

Figure 7: Padé approximation plot for the Hayward metric for $V(t,t')$ at $M = 1$, $r = 10$, $Q = 1$, $\Lambda = 1$.

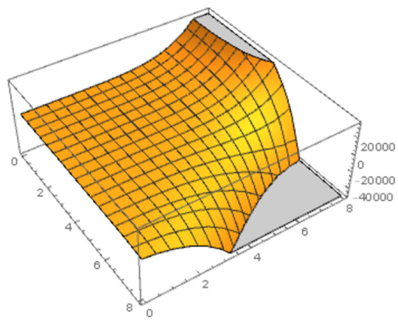

Figure 8: Plot of the surface of the function $V(x,x')$ as a function of coordinates and time for the Hayward metric at $M = 1$, $r = 1$, $l = 1$.

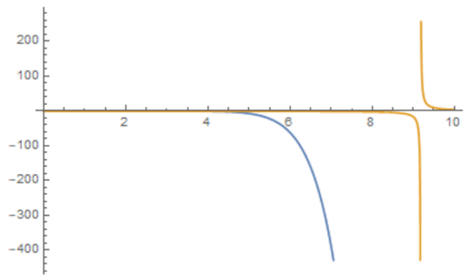

Figure 6: Plot of $V(t,t')$ function(blue line) and Padé approximation (orange line) for the Hayward metric at $M = 1$, $r = 1$, $Q = 1$, $\Lambda = 1$.

From these graphs, we observe a significant decrease in the number of Padé "singularities", which corresponds to the physical properties of this space: The Hayward metric has no singularities. The result of the Padé approximation (one singularity) can be explained by the insufficient accuracy of the calculations.

4.6 Green's function for the Bardeen metric

For the Bardeen metric one has [3, 21] $f(r) = 1 - \frac{2Mr^2}{(r^2+q_{BD}^2)^{3/2}}$, where q_{BD} is the magnetic charge.

We obtain a simplified residual part of Green's function for the Bardeen metric (for a spherically symmetric black hole, which is a source of the electric field and has no singularity but an event horizon):

$$V(t,t') = 4.419 \times 10^{-7} dt^8 - 1.117 \times 10^{-5} dt^6 \\ - 5.057 \times 10^{-4} dt^4 - 1.252 \times 10^{-2} dt^2 - 0.236.$$

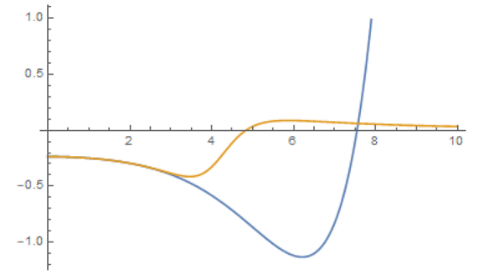

Figure 9: Plot of $V(t,t')$ function(blue line) and Padé approximation (orange line) for the Bardeen metric at $M = 1$, $r = 1$, $Q = 1$.

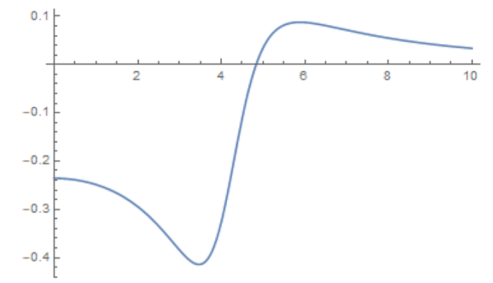

Figure 10: Padé approximation plot for the Bardeen metric for $V(t,t')$ at $M = 1$, $r = 1$, $Q = 1$.

In the resulting graphs, we observe extremum points for both functions. As for the residual part of the Green's function, we can say that these points correspond to the previously mentioned eigenvalues of the spectrum of the operator. The Padé approximation approaches them, but the accuracy is not high.

It seems quite reasonable to increase the computational complexity of the method in order to study the behavior of the residual Green's function. In this case, an increase of the singular points corresponding to the eigenvalues of the spectrum -

the quasi-normal modes - should be observed and the approximation should converge better to a power series $V(t, t')$.

5. Conclusion

In this paper, a consistent description of the algorithm that can be used to calculate the residual part of the Green's function was given, and calculations have been performed for six metrics corresponding to different types of black holes: the Schwarzschild metrics (symmetric, non-rotating and uncharged black hole), Reissner-Nordström metric (charged, non-rotating black hole), Schwarzschild-de Sitter metric (black hole with positive cosmological constant), Reissner-Nordström-de Sitter metric (when a constant electric charge is added to the cosmological term), Hayward metric (spherically symmetric, non-rotating, uncharged black hole having no singularity), Bardeen metric (for spherically symmetric black hole that is a source of an electric field and has no singularity but an event horizon). The consideration is based on the Hadamard-WKB method analogous to that in [10, 11]. Padé approximation plots are created for the residual parts of the Green's functions that indicate the presence of singularities.

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