MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
77, 2 (2025), 173–188
June 2025

research paper
оригинални научни рад
DOI: 10.57016/MV-OQYT5360

TOPOLOGICAL STUDY OF g-CONVERGENCE IN GENERALIZED 2-NORMED SPACES

Anirban Kundu, Tarapada Bag and Sk. Nazmul

Abstract. Some topological properties of generalized 2-normed (G2N) spaces have been studied in this article. The notion of g-convergence for sequences is introduced in general, and it is compared with the usual notion of convergence. It is shown that g-convergence is a more general idea, and under certain conditions g-convergence and convergence actually coincide. Using these concepts, a few fixed point theorems are developed.

1. Introduction

The need to generalize the notions of metric and norm has been felt since their formal introduction. In 1963, Gähler [5] proposed a 2-metric structure. He then introduced the notion of 2-norm in 1964 [6]. In his subsequent works [7–10] Gähler defined and studied n-metric and n-norm and considered metric and norm as 1-metric and 1-norm respectively. He claimed that these new concepts were generalizations of metric and norm. However, 2-metric and 2-norm satisfy some surprising properties that put researchers in a difficult situation. As a way out, B. C. Dhage [3] introduced D-metric spaces. The topological structure of the D-metric turned out to be defective [18]. To overcome this situation, two different modifications of the D-metric were developed, the G-metric and the S-metric. For a detailed understanding of these concepts, we recommend [13, 14, 16, 17, 19–21]. Although the expressions of G-metric and S-metric are different, the idea behind these two terms is the same. Both were introduced to calculate the distance between three points, keeping in mind the idea of the perimeter of a triangle. Chaipunya and Kumam [1] introduced $g - 3ps$-spaces to understand the possible notions of distance between three points from a general point of view. In another process, Czerwik [2] introduced the notion of b-metric spaces. Later, Mustafa et. al. [15] proposed the notion of b_2-metric spaces as a combination of the ideas of 2-metric and b-metric spaces.

2020 Mathematics Subject Classification: 46B20, 47A30, 46S99, 55M20, 54H25, 47H10
Keywords and phrases: Generalized 2-normed spaces; $g - 3ps$ spaces; G-metric space; S-metric spaces; g-convergence.

The notion of G-norm was introduced by K. A. Khan [11] in 2014. It was claimed that it is a generalization of the norm that induces a G-metric. But the research of Anirban Kundu, T. Bag and Sk. Nazmul [12] clearly indicates that there is no difference between the topologies of G-normed spaces and normed linear spaces. As an alternative, they proposed a new notion called generalized 2-norm.

Although this space is Hausdorff, it is not normable in general. Moreover, G2N-spaces induce $g-3ps$-spaces.

In this article, the notion of g-convergence in G2N-spaces is studied. It is found that g-convergence in general does not imply convergence. However, in some special cases these concepts coincide. Such conditions are studied using a new type of topology on G2N-spaces called g-topology. Finally, this topological structure is compared with the topology generated by G2N spaces. As an example, a special type of G2N that induces a G metric is introduced. Some fixed point theorems are proved using the ideas of g-convergence.

The rest of this article is divided into four sections. Section 2 contains the necessary definitions, ideas and results that already exist in the literature and are needed for the present article. Section 3 is the main part of this paper. In this section, the topological study of g-convergence and some related ideas are presented. Finally, Section 4 is the part where the ideas of g-convergence are applied to establish some fixed point theorems.

2. Preliminaries

The notion of G2N-spaces has been introduced in [12] as a generalization of normed linear spaces.

Definition 2.1 ([12]). Let X be a vector space over the field $\mathbb{K}$ (real or complex). A mapping $N : X \times X \to \mathbb{R}_{\geq 0}$ is called generalized 2-norm or G2N if it satisfies the following:

(GN1) $N(x, y) = 0$ if and only if $x = y = 0$,

(GN2) $N(\alpha x, \alpha y) = |\alpha| N(x, y)$ for every $x, y \in X$ and $\alpha \in \mathbb{K}$,

(GN3) There exist two positive numbers r and s such that $N(x-z, y-z) < s$ for all $x, y, z \in X$ satisfying $N(x, x)$, $N(y, y)$, $N(z, z) < r$.

The pair (X, N) is called a generalized 2-normed space or G2NS.

This space is closely related to $g-3ps$ space introduced in [1].

Definition 2.2 ([1]). Let X be a non-empty set. A function $g : X^3 \to \mathbb{R}_{\geq 0}$ is called a $g-3ps$ if:

(g1) $g(x, y, z) = 0$ if and only if $x = y = z$,

(g2) There exists some $r_0 > 0$ such that for each $x \in X$, $B_g(x, r_0) := \{y \in X : g(x, x, y) < r_0\}$ is bounded, i.e.

$$\sup_{a,b,c \in B_g(x,r_0)} g(a, b, c) < \infty.$$

The pair (X, g) is called a $g - 3ps$ space. The sets $B_g(x, r)$ are called open ball for the $g - 3ps$ space (X, g).

Chaipunya and Kumam [1] showed that $g - 3ps$-spaces are generalized forms of G-metric and S-metric spaces. For a better understanding, the definitions of these two notions are given below.

Definition 2.3 ([17]). Consider a non-empty set X. A function $G : X \times X \times X \to \mathbb{R}_{\geq 0}$ is called a G-metric on X if:
(G1) $G(x, y, z) = 0$ for $x, y, z \in X$ satisfying $x = y = z$,

(G2) $G(x, x, y) > 0$ for $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for every $x, y, z \in X$,

(G4) G is invariant under all permutations of (x, y, z),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.
The pair (X, G) is called a G-metric space.

Definition 2.4 ([20]). Consider a non-empty set X. A mapping $S : X \times X \times X \to \mathbb{R}_{\geq 0}$ is said to be an S-metric on X if
(S1) $S(x, y, z) = 0$ for $x, y, z \in X$ if and only if $x = y = z$,

(S2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $x, y, z, a \in X$.
The pair, (X, S) is called an S-metric space.

Following theorem is the bridge between G2N and $g - 3ps$-spaces.

Theorem 2.5 ([12]). *Consider a G2NS (X, N). The function $g : X \times X \times X \to \mathbb{R}_{\geq 0}$ defined by $g(x, y, z) := N(x - z, y - z)$ for every $x, y, z \in X$ is a $g - 3ps$ on X.*

Thus, any G2N can induce a $g - 3ps$. Open balls $B_N(x, r)$ in a G2N space are described in [12] as follows:

$$\begin{aligned} B_N(x, r) &:= B_g(x, r) \qquad \text{(Where } g \text{ is a } g - 3ps \text{ induced by } N) \\ &= \{y \in X : g(x, x, y) < r\} = \{y \in X : N(x - y, x - y) < r\}, \end{aligned}$$

for all $x \in X$ and $r > 0$.

Here is an important property of the open balls in a G2N space.

Proposition 2.6 ([12]). *If N is a G2N on X, then $B_N(x, r) = x + rB_N(0, 1)$ for each $x \in X$ and $r > 0$.*

The notion of maximal perimeter plays a critical role in the study of G2N-spaces.

Definition 2.7 ([12]). Let (X, N) be a G2NS. For $A \subseteq X$, $M(A) := \sup\{N(x - z, y - z) : x, y, z \in A\}$ is called maximal perimeter of A.

Remark 2.8 ([12]). If $M(A) < \infty$ then A is bounded, otherwise A is unbounded.

One can easily verify that every finite set is bounded in this sense. Moreover, the union of two bounded sets is also bounded.

Theorem 2.9 ([12]). *For a G2NS (X, N) we have:*
(i) $M(B_N(0,1)) < \infty$,

(ii) $M(B_N(x,r)) = M(B_N(0,r)) = rM(B_N(0,1))$ *for all* $x \in X$ *and* $r > 0$.

Remark 2.10. Theorem 2.9 establishes that every open ball in a G2N space is bounded. Moreover the expression of $M(B(x,r))$ is independent of x. So the following terminology makes sense $M_r := M(B_N(x,r))$ for all $r > 0$. Using this new notation Theorem 2.9 can be rephrased as: $M_r = rM_1$ for all $r > 0$ and so $\lim_{r\to 0} M_r = 0$. In another language, for any $\epsilon > 0$, there exists $s > 0$ such that $M_r < \epsilon$ whenever $r > s$.

The topology of G2NS was constructed in [12] using the set $\mathscr{B}_N$ of all open balls of (X, N) as follows.

Definition 2.11 ([12]). Let (X, N) be a G2NS. The topology of (X, N), denoted by τ_N, is defined as the smallest topology containing $\mathscr{B}_N$, the set of all open balls.

Following remark is a direct consequence of the Definition 2.11.

Remark 2.12 ([12]). $\mathscr{B}_N$ is a subbase of the topology τ_N.

A big family of examples of G2N-spaces can be constructed using the following proposition.

Proposition 2.13 ([12]). *For a periodic function* $\mathfrak{a} : [0, 2\pi) \to \mathbb{R}$ *with period* π, *satisfying* $\mathfrak{a}([0, 2\pi)) \subseteq [\epsilon, \delta]$ *for some* $\epsilon, \delta > 0$, *the function* $N : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ *defined by* $N(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = \frac{r_1+r_2}{2}\mathfrak{a}\left(\frac{\theta_1+\theta_2}{2}\right)$ *forms a G2NS,* $(\mathbb{R}^2, N)$.

A specific example of a G2N space in which this proposition is used is given in Section 3 (see Example 3.5).

Researchers have studied the notions of g-convergence and g-Cauchyness for various spaces. Chaipunya and Kumam [1] introduced these notions for $g - 3ps$-spaces as follows.

Definition 2.14 ([1]). Let X be a $g - 3ps$ space and $\{x_n\}$ be a sequence in it. Then,
(i) $\{x_n\}$ is said to be g-convergent if there is a point $x \in X$ such that for each $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $g(x, x, x_n) < \epsilon$ whenever $n \geq k$. In this case it is said that $\{x_n\}$ g-converges to x and is written as $x_n \xrightarrow{g} x$.

(ii) $\{x_n\}$ is said to be Cauchy if for any $\epsilon > 0$, there exixts $k \in \mathbb{N}$ such that $g(x_m, x_m, x_n) < \epsilon$ whenever $m, n \geq k$.

This article deals with g-convergence and related ideas from a topological point of view. For this purpose, a certain type of topological spaces is used, whose properties can be completely understood by convergent sequences. Spaces of this type are known as sequential spaces, which were introduced by S. P. Franklin [4].

Definition 2.15 ([4]). A topological space (X, τ) is said to be sequential if any set $F \subseteq X$ is closed if and only if any any convergent sequence in F converges in F.

In fact, we have another theorem to characterize sequential spaces.

THEOREM 2.16 ([4]). *A topological space* (X, τ) *is sequential if and only if for every open set* G*, each sequence in* X *converging in* G*, is eventually in* G*.*

EXAMPLE 2.17. Every first countable space is sequential. In particular, metric spaces are sequential.

3. Topological understanding of g-convergence

The notion of G-convergence was introduced by Mustafa and Sims [17] for sequences in G-metric spaces. Notions of a similar nature are also studied for S, D and $D*$-metric spaces. During the unification process of various generalized distance functions, Chaipunya and Kumam [1] reintroduced the same notion as g-convergence for $g-3ps$-spaces. The Definition 2.14 can be used in this context. In this section, we propose a possible analogy of g-convergence for G2N-spaces and find its topological meaning. Unless otherwise mentioned, X denotes a G2N space (X, N) in this section.

DEFINITION 3.1. Let X be a G2N space. A sequence $\{x_n\}$ in X is said to be g-convergent if there exists $x \in X$ such that for each $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $N(x - x_n, x - x_n) < \epsilon$ for all $n \geq k$. In such situation, x is said to be a g-limit of $\{x_n\}$ and is written as $x_n \xrightarrow{g} x$.

A few elementary properties are as follows.

PROPOSITION 3.2. *In a G2NS, g-limit of a g-convergent sequence is unique.*

Proof. Let $\{x_n\}$ be a g-convergent sequence with a g-limit x. If possible, let x' be another g-limit of $\{x_n\}$. Consider any $\epsilon > 0$. Then with the help of Remark 2.10 we may conclude that, there exists $r > 0$ such that $M_r < \epsilon$. Again, $\{x_n\}$ g-converges to x and x'. So, there exists $k \in \mathbb{N}$ such that

$$\begin{aligned} & N(x_n - x, x_n - x), N(x_n - x', x_n - x') < r && \text{whenever } n \geq k \\ \Rightarrow\ & x_n - x, x_n - x' \in B_N(0, r) && \text{whenever } n \geq k \\ \Rightarrow\ & N(x_k - x - x_k + x', x_k - x - x_k + x') < M_r < \epsilon && \text{(by Remark 2.10)} \\ \Rightarrow\ & N(x' - x, x' - x) < \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $N(x' - x, x' - x) = 0$ and so $x = x'$. □

PROPOSITION 3.3. *Every g-convergent sequence in a G2NS is bounded.*

Proof. Let $\{x_n\}$ be a sequence in X which g-converges to a point $x \in X$. Then, for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$\begin{aligned} & N(x_n - x, x_n - x) < \epsilon && \text{whenever } n \geq k \\ \Rightarrow\ & x_m - x, x_n - x, x_p - x \in B_N(0, \epsilon) && \text{whenever } m, n, p \geq k \\ \Rightarrow\ & N(x_m - x - x_p + x, x_n - x - x_p + x) < M_\epsilon && \text{whenever } m, n, p \geq k \end{aligned}$$

$$\Rightarrow\ N(x_m - x_p, x_n - x_p) < M_\epsilon \qquad\qquad \text{whenever}\ \ m,n,p \geq k.$$

So, $A = \{x_n : n \geq k\}$ is a bounded set. Again, $B = \{x_n : n < k\}$ is a finite set. So, it is also bounded. Thus, the set $A \cup B$, which is the collection of all the elements of the sequence $\{x_n\}$, is bounded. □

REMARK 3.4. Chaipunya and Kumam [1] stated that for sequences in $g - 3ps$-spaces g-convergence and convergence are two different notions. Indeed, convergence implies g-convergence, but not vice versa. Since a G2N space is also a $g - 3ps$-space, one can conclude that a sequence $\{x_n\}$ in a G2N space X converges to x, which implies that $\{x_n\}$ is g-convergent to x. But the converse is not true for G2N-spaces either. We illustrate this with an example.

EXAMPLE 3.5. Consider a function $\mathfrak{a} : [0, 2\pi) \to \mathbb{R}$ defined by

$$\mathfrak{a}(\theta) = \begin{cases} \frac{\pi}{2\pi-\theta} & \text{if } \theta \in [0,\pi) \\ \frac{\pi}{3\pi-\theta} & \text{if } \theta \in [\pi, 2\pi). \end{cases}$$

The Proposition 2.13 leads to the conclusion that $\mathfrak{a}$ defines a G2N on $\mathbb{R}^2$. We are interested to understand the topology τ_N in more details. For this purpose, consider the set $B_N(0,1)$. We have,

$$\begin{aligned} B_N(0,1) &= \{re^{i\theta} : N(re^{i\theta}, re^{i\theta}) < 1\} = \{re^{i\theta} : r\mathfrak{a}(\theta) < 1\} \\ &= \Big\{re^{i\theta} : r < \frac{2\pi-\theta}{\pi}, 0 \leq \theta < \pi\Big\} \cup \Big\{re^{i\theta} : r < \frac{3\pi-\theta}{\pi}, \pi \leq \theta < 2\pi\Big\}. \end{aligned}$$

One can check that, $B_N((0,0),1) \cap B_N((3,0),1) = \{(x,0) : 1 < x < 2\}$. In fact, any set of the form $S((a,c),(b,c)) = \{(x,c) : a < x < b\}$ can be written as intersection two open balls and so is an open set. Hence, $\mathscr{B} = \{((a,c),(b,c)) : a,b,c \in \mathbb{R}\} \subseteq \tau_N$. Observe that, the elements of $\mathscr{B}$ are the open segments on straight lines parallel to x-axis. Moreover, any finite intersection of open balls can be written as a union of some elements of $\mathscr{B}$. Thus, $\mathscr{B}$ is a basis of τ_N. If we consider a metric, $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d((x_1,y_1),(x_2,y_2)) = \begin{cases} |x_1 - x_2| & \text{if } y_1 = y_2 \\ |x_1| + |x_2| + 1 & \text{if } y_1 \neq y_2 \end{cases},$$

then, the $\mathscr{B}$ is a subset of open balls in $(\mathbb{R}^2, d)$. Moreover, all other open balls of $(\mathbb{R}^2, d)$ can be written as union of elements of $\mathscr{B}$. Hence, the topology of $(\mathbb{R}^2, d)$ is nothing but (X, τ_N). Thus, (X, τ_N) is metrizable.

Now, consider the sequence $\{x_n\}$ defined by

$$x_n = \Big(0, \frac{1}{n}\Big) \ \text{ for each } \ n \in \mathbb{N}.$$

Consider the open set $((-1,0),(1,0))$ containing $(0,0)$. Then, for every $n \in \mathbb{N}$,

$$x_n = \Big(0, \frac{1}{n}\Big) \notin ((-1,0),(1,0)).$$

However,

$$N(x_n - (0,0), x_n - (0,0)) = N\Big(\Big(0, \frac{1}{n}\Big), \Big(0, \frac{1}{n}\Big)\Big) = \frac{1}{n} \cdot \frac{\pi}{2\pi - \frac{\pi}{2}} = \frac{1}{3n}.$$

Hence, as $n \to \infty$, $N(x_n - (0,0), x_n - (0,0)) \to 0$. Therefore, $x_n \nrightarrow (0,0)$ but $x_n \xrightarrow{g} (0,0)$.

The above example clearly shows the difference between convergence and g-convergence of sequences in G2N-spaces. With this difference in mind, let us examine g-convergence from a topological point of view. We investigate the possibility of a topology for which these two notions coincide. First, we define g-open sets.

Definition 3.6. A subset G of a G2NS X is said to be g-open if for any $x \in G$ and any sequence $\{x_n\}$ in X g-converging to x, there exists $k \in \mathbb{N}$ such that $x_n \in G$ whenever $n \geq k$.

The relation between open sets and g-open sets is an interesting point to study. We discuss on this after introducing a new kind of topology.

Proposition 3.7. *For a G2N space (X, N), the set τ_g of all g-open sets forms a topology on X.*

Proof. Let (X, N) be a G2N space and τ_g be the set of all g-open sets. Then, by definition, empty set ϕ and X are in τ_g. Now consider an arbitrary family $\{G_\lambda : \lambda \in \Lambda\} \subseteq \tau_g$. Take any $x \in \bigcup_{\lambda \in \Lambda} G_\lambda$. Then $x \in G_{\lambda'}$ for some $\lambda' \in \Lambda$. So, for any sequence $\{x_n\}$ in X g-converging to x, there exists $k \in \mathbb{N}$ such that $x_n \in G_{\lambda'}$ whenever $n \geq k$. But $G_{\lambda'} \subseteq \bigcup_{\lambda \in \Lambda} G_\lambda$. Thus, $x_n \in \bigcup_{\lambda \in \Lambda} G_\lambda$ whenever $n \geq k$ and so $\bigcup_{\lambda \in \Lambda} G_\lambda \in \tau_g$.

Finally, consider $G_1, G_2, \ldots, G_m$ from τ_g. We need to show that $\bigcap_{i=1}^{m} G_i \in \tau_g$. For this, take any $x \in \bigcap_{i=1}^{m} G_i$ and any sequence $\{x_n\}$ in X g-converging to x. Thus, for each $i \in \{1, 2, \ldots, m\}$ we have $x_n \xrightarrow{g} x \in G_i \in \tau_g \Rightarrow$ there exists $k_i \in \mathbb{N}$ such that $x_n \in G_i$ whenever $n \geq k_i$. Take $k = \max\{k_i : i = 1, 2, \ldots, m\}$. Then, for each $i \in \{1, 2, \ldots, m\}, x_n \in G_i$ whenever $n \geq k \Rightarrow x_n \in \bigcap_{i=1}^{m} G_i$ whenever $n \geq k$.

Hence, $\bigcap_{i=1}^{m} G_i \in \tau_g$. Therefore, τ_g forms a topology on X. $\square$

Remark 3.8. We propose to call this topology τ_g as "g-topology". Note that, for the topological space (X, τ_g), the open sets are the g-open sets. Since there is another topology τ_N into play, we prefer to call the open sets in (X, τ_N) as N-open.

Following theorem gives an alternative description of g-open sets.

Theorem 3.9. *Let X be a G2NS. $G \subseteq X$ is g-open if and only if for each $x \in G$ there exists $r > 0$ such that $B_N(x, r) \subseteq G$.*

Proof. Suppose X be a G2NS and $G \subseteq X$ be a g-open set. Take any $x \in G$. We need to show the existence of a $r > 0$ for which $B_N(x, r) \subseteq G$. If no such $r > 0$ exists, then for any $\frac{1}{n}$, where $n \in \mathbb{N}$, we have $B_N\left(x, \frac{1}{n}\right) \nsubseteq G$ there exists $x_n \in B_N\left(x, \frac{1}{n}\right)$ such that $x_n \notin G \Rightarrow N(x_n - x, x_n - x) < \frac{1}{n}$ and $x_n \notin G$.

In this way, we have a sequence $\{x_n\}$ such that $x_n \xrightarrow{g} x$ but $x_n \notin G$ for each $n \in \mathbb{N}$. This contradicts the g-openness of G. Hence, there exists $r > 0$ for which $B_N(x,r) \subseteq G$.

Conversely, let $G \subseteq X$ such that for each $x \in X$, there exists $r > 0$ for which $B_N(x,r) \subseteq G$. Take any sequence $\{x_n\}$ in X so that $x_n \xrightarrow{g} x$. Then, there exists $k \in \mathbb{N}$ for which $N(x_n - x, x_n - x) < r$ whenever $n \geq k \Rightarrow$ $x_n \in B_N(x,r)$ whenever $n \geq k \Rightarrow$ $x_n \in G$ whenever $n \geq k$. Thus, G is a g-open set. □

With reference to the above theorem, we have an alternative definition of g-topology as follows.

Definition 3.10. In a G2NS (X,N), the g-topology τ_g is defined as follows

$$\tau_g := \{G \subseteq X : \text{for each } x \in G \text{ there exists } r > 0 \text{ such that } B_N(x,r) \subseteq G\}.$$

The idea of g-topology originated from the concept g-convergence. Now we move back to discuss the notion of g-convergence under the framework of g-topology.

Lemma 3.11. *For a G2NS (X,N), $x \in X$ and a sequence $\{x_n\}$ in X, $x_n \to x$ in (X,τ_g) if $x_n \xrightarrow{g} x$ in (X,N).*

Proof. Suppose $\{x_n\}$ be a sequence in X and $x \in X$ such that $x_n \xrightarrow{g} x$. Take any $G \in \tau_g$ containing x. Then, there exists $r > 0$ such that $x \in B_N(x,r) \subseteq G$. Now,

$$\begin{aligned} x_n \xrightarrow{g} x \text{ in } (X,N) &\Rightarrow N(x_n - x, x_n - x) \to 0 \text{ as } n \to \infty \\ &\Rightarrow \text{there exists } k \in \mathbb{N} \text{ such that } N(x_n - x, x_n - x) < r \text{ whenever } n \geq k \\ &\Rightarrow \text{there exists } k \in \mathbb{N} \text{ such that } x_n \in B_N(x,r) \text{ whenever } n \geq k \\ &\Rightarrow \text{there exists } k \in \mathbb{N} \text{ such that } x_n \in G \text{ whenever } n \geq k \\ &\Rightarrow x_n \to x \text{ in } (X,\tau_g). \end{aligned}$$

□

Remark 3.12. Combining Remark 3.4 and Lemma 3.11 we may conclude that, for a point x and a sequence $\{x_n\}$ in (X,N), $x_n \to x$ in $(X,\tau_N) \Rightarrow x_n \xrightarrow{g} x$ in $(X,N) \Rightarrow$ $x_n \to x$ in (X,τ_g). So, if τ_N and τ_g coincide under certain condition, then all the notions of convergence coincide.

We have the following proposition regarding the relation between τ_N and τ_g.

Proposition 3.13. *For a G2NS (X,N), we have τ_N is finer than τ_g.*

Proof. Suppose (X,N) be a G2NS. Take any g-open set G. Then by Definition 3.10, for each $x \in G$, we have $r_x > 0$ such that $B_N(x,r_x) \subseteq G$. Thus, $G = \bigcup_{x \in G} B_N(x,r_x)$. Now, each $B_N(x,r_x) \in \tau_N$. Thus, $G \in \tau_N$. Since G is an arbitrarily chosen g-open set, therefore $\tau_g \subseteq \tau_N$. □

Remark 3.14. For the above proposition, equality relation does not hold in general. As an example, consider the G2NS defined in Example 3.5. There, the sequence $x_n = \left(0, \frac{1}{n}\right)$ does not converge to $(0,0)$. But, g-converges to $(0,0)$. Hence, $\tau_N \neq \tau_g$.

Here we examine the case when $\tau_N = \tau_g$ in detail and conclude this section. Recall that Definition 3.10 gives a description of g-open sets. From this definition, we can conclude that the open balls $B_N(x, r)$ play a crucial role in the description of g-open sets. Indeed, g-open sets can be written as a union of open balls. However, open balls are not g-open sets in general. The following theorem successfully binds all these issues as a necessary and sufficient criterion.

THEOREM 3.15. *For a G2NS* (X, N)*, the followings are equivalent:*
(I) $\mathscr{B}_N \subseteq \tau_g$*, (II)* $\mathscr{B}_N$ *is a basis of* τ_g*, (III)* $\tau_g = \tau_N$.

Proof. (I) $\Leftrightarrow$ (II): Suppose $\mathscr{B}_N \subseteq \tau_g$. So, the open balls are g-open sets. Thus, from the definition of g-open sets it follows that $\mathscr{B}_N$ is a basis of τ_g.

On the other hand, basis of a topology is always a subset of that topology. Thus, $\mathscr{B}_N$ is a basis of τ_g and thus $\mathscr{B}_N \subseteq \tau_g$.

(I) $\Leftrightarrow$ (III): Suppose $\mathscr{B}_N \subseteq \tau_g$. Now,

$$\begin{aligned} \mathscr{B}_N \subseteq \tau_g &\Rightarrow \tau_N \subseteq \tau_g \quad (\text{since } \mathscr{B}_N \text{ is a subbasis of } \tau_N) \\ &\Rightarrow \tau_N = \tau_g \quad (\text{since } \tau_g \subseteq \tau_N). \end{aligned}$$

Conversely, if $\tau_N = \tau_g$, then $\mathscr{B}_N \subseteq \tau_N = \tau_g$. □

REMARK 3.16. The possibility of equality for τ_N and τ_g is discussed in Theorem 3.15. From this it can be concluded that the notion of g-convergence can be considered as the well-known notion of convergence for sequences if one of the three conditions mentioned in Theorem 3.15 hold.

Topological properties are easier to understand if sequences play the key role. We conclude this section with such discussion for G2NS.

THEOREM 3.17. *For a G2NS* (X, N)*, if* $\tau_N = \tau_g$*, then the topology is sequential.*

Proof. For instance, let $\tau_N = \tau_g = \tau$. Now, take any closed set F. Choose any sequence $\{x_n\}$ in F. Suppose, x be such that $x_n \to x$. Remark 3.12 assures that $x_n \xrightarrow{g} x$. Since, g-limit of a sequence is unique, we conclude that x is unique and so there is no other limit of $\{x_n\}$. Thus, it is enough to show that $x \in F$.

If $x \notin F$, then $x \in X \setminus F$. Now, $X \setminus F$ is an open set in (X, τ_g). So, by Definition 3.6 we have $k \in \mathbb{N}$ such that $x_n \in X \setminus F$ whenever $n \geq k$. Thus, $x_k \in X \setminus F$ and so $x_k \notin F$. This contradicts our assumption. Hence, $x \in F$. Therefore, τ is a sequential space. □

4. Further study on g-convergence and some related fixed point theorems

In this section we study some properties of g-convergence that were not covered in Section 3. Some related notions, such as g-Cauchy, g-quasi-Cauchy sequences, are also introduced here. The notions of g-complete and g-compact spaces are also briefly introduced. We start with definitions of g-Cauchy and g-quasi-Cauchy sequences.

Definition 4.1. In a G2NS X, a sequence $\{x_n\}$ is said to be g-Cauchy if for each $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $N(x_m - x_n, x_m - x_n) < \epsilon$ whenever $m, n \geq k$.

Definition 4.2 (g-Quasi-Cauchyness). A sequence $\{x_n\}$ in a G2NS X is said to be g-quasi-Cauchy if for each $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $N(x_{n+1} - x_n, x_{n+1} - x_n) < \epsilon$ whenever $n \geq k$.

An important result related to Definition 4.2 is as follows.

Lemma 4.3. *If a sequence $\{x_n\}$ in a G2NS X is g-quasi-Cauchy and the subsequence $\{x_{n_k}\}$ g-converges to x,then $\{x_{n_k+1}\}$ also g-converges to x.*

Proof. Let $\epsilon > 0$ be arbitrary. Then by Remark 2.10 there exists $r > 0$ such that $M_r < \epsilon$.

Given that $\{x_{n_k}\}$ g-converges to x. So for $r > 0$ there exists $N_1 \in \mathbb{N}$ such that, whenever $n_k \geq N$ we have $N(x_{n_k} - x, x_{n_k} - x) < r$. On the other hand $\{x_n\}$ is g-quasi-Cauchy. Thus there exists $N_1 \in \mathbb{N}$ such that $N(x_{n_k+1} - x_{n_k}, x_{n_k+1} - x_{n_k}) < r$ for all $n_k \geq N_2$.

Choose $N = \max\{N_1, N_2\}$. Then for every $n_k \geq N$ we have: $N(x_{n_k} - x, x_{n_k} - x) < r$ and $N(x_{n_k+1} - x_{n_k}, x_{n_k+1} - x_{n_k}) < r$. So, $N(x_{n_k+1} - x, x_{n_k+1} - x) < M_r < \epsilon$ for all $n_k \geq N$. Hence $\{x_{n_k+1}\}$ g-converges to x. □

The notions of g-Cauchyness and g-convergence are linked. The following results prove this, the proof is obvious.

Proposition 4.4. *In a G2NS, any subsequence of a g-convergent sequence is g-convergent and the g-limits are same.*

Proposition 4.5. *In a G2NS, every g-convergent sequence is g-Cauchy.*

Proof. Consider a g-convergent sequence $\{x_n\}$. Choose any $\epsilon > 0$. Then by Remark 2.10 there exists $r > 0$ such that $M_r < \epsilon$.

Since $\{x_n\}$ is g-convergent, there exists $k \in \mathbb{N}$ such that $N(x - x_n, x - x_n) < r$ whenever $n \geq k$.

So for $m, n \geq k$ we have $N(x - x_m, x - x_m) < r$ and $N(x - x_n, x - x_n) < r$. Therefore $N(x_m - x_n, x_m - x_n) < M_r < \epsilon$ for $m, n \geq k$. Hence $\{x_n\}$ is g-Cauchy. □

Converse of Proposition 4.5 is not always true. We have the following counterexample.

Example 4.6. Consider $C[0, 1]$, the set of all real valued continuous functions defined on $[0, 1]$. Then it is already an established fact that, the norm $\|\cdot\|_2$ on $C[0, 1]$ defined by

$$\|f\|_2 := \Big(\int_0^1 (f(x))^2 dx \Big)^{1/2} \quad \text{for every} \quad f \in C[0, 1],$$

gives an incomplete normed linear space. Now, we define a G2N on $C[0, 1]$ as follows,

$$\|f, g\| := \|f - g\|_2 + \|f\|_2 \quad \text{for every} \quad f, g \in C[0, 1].$$

For this G2N,

a sequence $\{x_n\}$ is g-Cauchy in $(C[0,1], \|\cdot,\cdot\|)$

$\Leftrightarrow$ for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $\|x_m - x_n, x_m - x_n\| < \epsilon$ whenever $m, n \geq k$

$\Leftrightarrow$ for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $\|x_m - x_n\|_2 < \epsilon$ whenever $m, n \geq k$

$\Leftrightarrow$ $\{x_n\}$ is Cauchy in $(C[0,1], \|\cdot\|_2)$.

Using similar argument it can be proved that, a sequence $\{x_n\}$ g-converges to x if and only if it converges to x in $(C[0,1], \|\cdot\|_2)$. These two armuments clearly establishes that, if every g-Cauchy sequence g-converges in $(C[0,1], \|\cdot,\cdot\|)$, then $(C[0,1], \|\cdot\|_2)$ is complete, which is a contradiction. Thus there are g-Cauchy sequences which are not g-converges in $(C[0,1], \|\cdot,\cdot\|)$.

However, some added condition proves the converse of Proposition 4.5.

Theorem 4.7. *If a g-Cauchy sequence has a subsequence g-converging to $x \in X$, then it g-converges to x.*

Proof. Let $\epsilon > 0$ be given. Then by Remark 2.10 there exists $r > 0$ such that $M_r < \epsilon$. Now for $r > 0$ there exists $k \in \mathbb{N}$ such that for all $m, n, n_k \geq N$ we have $N(x_m - x_n, x_m - x_n) < r$ and $N(x_{n_k} - x, x_{n_k} - x) < r$. So, $N(x_{n_k} - x_n, x_{n_k} - x_n), N(x_{n_k} - x, x_{n_k} - x) < r$ for all $n_k, n \geq N$. Therefore, $N(x_n - x, x_n - x) < M_r < \epsilon$ for all $n \geq N$ and so $x_n \overset{g}{\to} x$. □

However, there are spaces where one does not need to add such condition. Such spaces are called g-complete.

Definition 4.8 (g-completeness). A non-empty subset S of a G2NS (X, N) is called g-Complete if every g-Cauchy sequence in S is g-converges in S.

Definition 4.9 (Contraction Mapping). Let (X, N) be a G2NS and $S \subseteq X$ be nonempty. A mapping $T : S \to S$ is called a contraction on S if there exists $\alpha \in (0,1)$ such that the following holds for every $x, y \in S$, $N(T(x) - T(y), T(x) - T(y)) \leq \alpha N(x - y, x - y)$.

Theorem 4.10 (Contraction Theorem - Banach Type). *Let (X, N) be a G2NS and $S \subseteq X$ be nonempty. T be a contraction on S. If for some $x_0 \in S$ the sequence $\{x_n\}$ defined by $x_n := T^n(x_0)$ $\forall n \in \mathbb{N}$, has a subsequence $\{x_{n_k}\}$ g-converging in S, then T has a unique fixed point in S. Moreover, $\{x_{n_k}\}$ g-converges to that unique fixed point.*

Proof. For all $n \in \mathbb{N}$ we have

$$N(x_{n+1} - x_n, x_{n+1} - x_n) = N(T(x_n) - T(x_{n-1}, T(x_n) - T(x_{n-1}) \\ \leq \alpha N(x_n - x_{n-1}, x_n - x_{n-1}) \ldots \leq \alpha^n N(x_1 - x_0, x_1 - x_0).$$

Now, $\alpha \in [0,1)$ implies that $N(x_{n+1} - x_n, x_{n+1} - x_n) \to 0$ as $n \to \infty$.
So $\{x_n\}$ is g-quasi-Cauchy.

Let $\{x_{n_k}\}$ be a g-convergent subsequence of $\{x_n\}$. Suppose $x_{n_k} \overset{g}{\to} x$. Now, $N(T(x) - x_{n_k+1}, T(x) - x_{n_k+1}) < \alpha N(x - x_{n_k}, x - x_{n_k})$ for all $n_k \in \mathbb{N}$. However,

$x_{n_k} \xrightarrow{g} x$. So $N(x - x_{n_k}, x - x_{n_k}) \to 0$ as $n_k \to \infty$. So by Sandwich theorem we can say that $N(T(x) - x_{n_k+1}, T(x) - x_{n_k+1}) \to 0$ as $n_k \to \infty$ i.e. $x_{n_k+1} \xrightarrow{g} T(x)$.

On the other hand, Lemma 4.3 concludes that $x_{n_k+1} \xrightarrow{g} x$.

Since a G2NS is Hausdorff, we can say that $T(x) = x$, i.e. x is a fixed point for T. If x is not the unique fixed point, then there exists another point $y \in S$ such that $T(y) = y$. Now, $N(x - y, x - y) = N(T(x) - T(y), T(x) - T(y)) \leq \alpha N(x - y, x - y)$. According to our assumption, $x \neq y$ and so $N(x - y, x - y) \neq 0$. Thus, $\alpha \geq 1$, which is a contradiction.

Therefore, X contains a unique fixed point for T. □

Remark 4.11. Theorem 4.10 can be seen as a general theorem for functions satisfying the Banach Type contraction. We have simpler results for some special cases as follows.

Definition 4.12. Let (X, N) be a G2NS. A nonempty subset S of X is said to be a unitary disc if its maximal perimeter is less than or equal to 1.

Theorem 4.13. *Let X be a G2NS and S be a g-complete unitary disc in X. Let T be a contraction on S. Then T has a unique fixed point.*

Proof. Consider any $x_0 \in X$ and construct the iterative sequence $\{x_n\}$ as: $x_n := T^n(x_0)\ \forall n \in \mathbb{N}$. Then, proceeding similarly as Theorem 4.10 we have, for each $n \in \mathbb{N}$ $N(x_{n+1} - x_n, x_{n+1} - x_n) \leq \alpha^n N(x_1 - x_0, x_1 - x_0)$. Let, $r = \alpha^n N(x_1 - x_0, x_1 - x_0)$. Then,

$$N(x_{n+1} - x_n, x_{n+1} - x_n) \leq r. \tag{1}$$

Now,

$$\begin{aligned} N(x_{n+2} - x_{n+1}, x_{n+2} - x_{n+1}) &= \alpha N(T(x_{n+1}) - T(x_n), T(x_{n+1}) - T(x_n)) \\ &\leq \alpha N(x_{n+1} - x_n, x_{n+1} - x_n) = \alpha r < r. \end{aligned} \tag{2}$$

From equations (1) and (2) we have,

$$\begin{aligned} &N(x_{n+2} - x_n, x_{n+2} - x_n) \\ &= N((x_{n+2} - x_{n+1}) + (x_{n+1} - x_n), (x_{n+2} - x_{n+1}) + (x_{n+1} - x_n)) \leq M_r = rM_1 \leq r. \end{aligned}$$

Proceeding in this way iteratively, for any $p \in \mathbb{N}$ we get

$$N(x_{n+p} - x_n, x_{n+p} - x_n) \leq r. \tag{3}$$

Now, as $n \to \infty$, $r = \alpha^n N(x_1 - x_0, x_1 - x_0) \to 0$ and r is independent of p. Equation (3) leads to the conclusion that $\{x_n\}$ is a Cauchy sequence. Thus, the g-Completeness of S proves that $\{x_n\}$ g-converges in S. Let, $x_n \xrightarrow{g} x$ as $n \to \infty$. Since for any $n \in \mathbb{N}$,

$$N(x_{n+1} - T(x), x_{n+1} - T(x)) = N(T(x_n) - T(x), T(x_n) - T(x)) \leq \alpha N(x_n - x, x_n - x),$$

therefore, $x_{n+1} \xrightarrow{g} T(x)$ as $n \to \infty$. Since limit of a sequence is unique, we have $x = T(x)$. Uniqueness of the fixed point follows similarly as Theorem 4.10. □

Definition 4.14. Let X be a G2NS and $S \subseteq X$. A function $T : S \to S$ is said to be contractive if for every $x, y \in S$ with $x \neq y$, the following holds:

$$N(T(x) - T(y), T(x) - T(y)) < N(x - y, x - y).$$

The fixed point theorem for contractive mappings require the following definition.

Definition 4.15. A G2NS X is said to be g-continuous if for any two sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \xrightarrow{g} x$ and $y_n \xrightarrow{g} y$ we have, $N(x_n, y_n) \to N(x, y)$.

Theorem 4.16 (Contractive Theorem - Edelstein Type). *Let (X, N) be a g-continuous G2NS and T be a contractive mapping on $S \subseteq X$. If for some $x_0 \in S$, the iterative sequence $\{x_n\}$ defined as $x_n := T^n(x_0)$ for each $n \in \mathbb{N}$ has a g-convergent subsequence converging in S, then T has a unique fixed point.*

Proof. Let $T : S \to S$ be a contractive mapping on $S \subseteq X$. Suppose $x_0 \in S$ be such that the iterative sequence $\{x_n\}$ defined by $x_n := T^n(x_0)$ has a convergent subsequence. Let $\{x_{n_i}\}$ be such subsequence of $\{x_n\}$ converging to a point ξ in S. We are interested to find a fixed point for T. If $\{x_n\}$ has two consecutive terms of equal value, then we have:

$$x_m = x_{m+1} \text{ for some } m \in \mathbb{N} \ \Rightarrow\ T(x_m) = x_m \ \Rightarrow\ x_m \text{ is a fixed point for } T.$$

If no two consecutive terms of $\{x_n\}$ are equal, i.e. $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$, then we claim that ξ is a fixed point. If not, then $T(\xi) \neq \xi$. So by the contractive condition of T,

$$N(T(\xi) - T^2(\xi), T(\xi) - T^2(\xi)) < N(\xi - T(\xi), \xi - T(\xi)). \tag{4}$$

Now, consider the sequence $\{a_n\}$, where the terms are defined as

$$a_n := N(T^n(x_0) - T^{n+1}(x_0), (T^n(x_0) - T^{n+1}(x_0)).$$

Clearly, for any $n \in \mathbb{N}$,

$$\begin{aligned} a_{n+1} &= N(T^{n+1}(x_0) - T^{n+2}(x_0), (T^{n+1}(x_0) - T^{n+2}(x_0)) \\ &< N(T^n(x_0) - T^{n+1}(x_0), (T^n(x_0) - T^{n+1}(x_0)) = a_n. \end{aligned}$$

Thus, $\{a_n\}$ is a decreasing sequence. Now,

$x_{n_i} \xrightarrow{g} \xi \ \Rightarrow\ N(x_{n_i} - \xi, x_{n_i} - \xi) \to 0$

$\Rightarrow$ For any $\epsilon > 0$ there exists $m \in \mathbb{N}$ such that $N(x_{n_i} - \xi, x_{n_i} - \xi) < \epsilon$ whenever $i \geq m$

$\Rightarrow N(x_{n_i+1} - T(\xi), x_{n_i+1} - T(\xi)) < N(x_{n_i} - \xi, x_{n_i} - \xi) < \epsilon$ whenever $i \geq m$

$\Rightarrow x_{n_i+1} \xrightarrow{g} T(\xi)$.

Similarly, $x_{n_i+2} \xrightarrow{g} T^2(\xi)$. Thus, $x_{n_i} - x_{n_i+1} \xrightarrow{g} \xi - T(\xi)$ and $x_{n_i+1} - x_{n_i+2} \xrightarrow{g} T(\xi) - T^2(\xi)$. Continuity of N leads to the following,

$$a_{n_i}| = N(x_{n_i} - x_{n_i+1}, x_{n_i} - x_{n_i+1}) \xrightarrow{g} N(\xi - T(\xi), \xi - T(\xi))$$

and

$$a_{n_i+1} = N(x_{n_i+1} - x_{n_i+2}, x_{n_i+1} - x_{n_i+2}) \xrightarrow{g} N(T(\xi) - T^2(\xi), T(\xi) - T^2(\xi)).$$

Thus, $N(\xi - T(\xi), \xi - T(\xi))$ is a subsequential limit of $\{a_n\}$.

Choose any $k \in \mathbb{N}$ arbitrarily. Then, for any $n > n_k + 1$ we have, $a_n < a_{n_k+1}$ So, any subsequential limit of $\{a_n\}$ is also less than a_{n_k+1}. Therefore, $N(\xi - T(\xi), \xi - T(\xi)) < a_{n_k+1}$. Since the choice of k is arbitrary, we have a subsequence $\{a_{n_k+1}\}$ whose terms are greater than $N(\xi - T(\xi), \xi - T(\xi))$. So, the limit of $\{a_{n_k+1}\}$, which is $N(T(\xi) - T^2(\xi), T(\xi) - T^2(\xi))$, satisfies the following condition, $N(\xi - T(\xi), \xi - T(\xi)) \leq N(T(\xi) - T^2(\xi), T(\xi) - T^2(\xi))$. This contradicts (4). Hence, $T(\xi) = \xi$. □

Finally we introduce the notion of g-compactness and study fixed point theorems in such spaces.

Definition 4.17. In a G2NS (X, N), $S \subseteq X$ is said to be g-compact if every sequence in S has a g-convergent subsequence converging in S.

Remark 4.18. For a g-compact set, every sequence has a convergent subsequence. Hence, in particular for Theorem 4.10 and Theorem 4.16, the iterative sequence always has a convergent subsequence. Thus, we have the following result.

Theorem 4.19. *Let (X, N) be a continuous G2NS and $S \subseteq X$ be a g-compact set. Let $T : S \to S$ be a mapping satisfying the following condition for every $x \in S$,*

$$N(T(x) - T(y), T(x) - T(y)) \leq N(x - y, x - y).$$

Then, T has a unique fixed point.

We end this section with an example justifying the fixed point theory studied here.

Example 4.20. Consider the G2NS defined in Example 3.5. Consider a mapping

$$T(re^{i\theta}) = \begin{cases} \frac{r}{5} e^{i(\theta+\pi)} & \text{if } \theta \in [0, \pi) \\ \frac{r}{5} e^{i(\theta-\pi)} & \text{if } \theta \in [\pi, 2\pi). \end{cases}$$

Then, T is actually a linear transformation and

$$\begin{aligned} N(T(re^{i\theta}), T(re^{i\theta})) &= N(\frac{r}{5} e^{i(\theta+\pi)}, \frac{r}{5} e^{i(\theta+\pi)}) \\ &= \frac{r}{5} \mathfrak{a}(\theta + \pi) = \frac{r}{5} \mathfrak{a}(\theta) = \frac{1}{5} N(re^{i\theta}, re^{i\theta}). \end{aligned}$$

Thus,

$$\begin{aligned} &N(T(r_1 e^{i\theta_1}) - T(r_2 e^{i\theta_2}), T(r_1 e^{i\theta_1}) - T(r_2 e^{i\theta_2})) \\ &= N(T(r_1 e^{i\theta_1} - r_2 e^{i\theta_2}), T(r_1 e^{i\theta_1} - r_2 e^{i\theta_2})) = \frac{1}{5} N(r_1 e^{i\theta_1} - r_2 e^{i\theta_2}, r_1 e^{i\theta_1} - r_2 e^{i\theta_2}). \end{aligned}$$

This proves that T is a contractive mapping. Consider the iterative sequence with $x_0 = e^{i.0}$ Then, the iterative sequence $\{x_n\}$ converges to $(0, 0)$. Hence, T has a unique fixed point $(0, 0)$.

5. Conclusion

The notion of g-convergence has been studied by many researchers for a long time. It has been found that this notion is similar but not identical to the notion of convergence. In fact, there has been no adequate study to demonstrate these notions topologically. In this paper, this task is solved by the authors by introducing a new type of topology, the g-topology. As a result, some fixed point theorems are also established.

Acknowledgement. The authors are grateful to the anonymous reviewer for his/her valuable suggestions and comments in rewriting the article in the present form. The authors are also thankful to the Editor-in-Chief of the journal (Matematicki Vesnik) for her valuable comments.

The authors are grateful to Department of Mathematics, Siksha-Bhavana, Visva-Bharati; Department of Basic Sciences and Humanities, Ramkrishna Mahato Government Engineering College and Department of Mathematics, Kazi Nazrul Univeristy for allowing this research. The infrastructure and other kinds of encouragements provided by those Institutions were much needed for this work.

References

[1] P. Chaipunya, P. Kumam, *On the distance between three arbitrary points*, J. Funct. Spaces, (2013).

[2] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostrav. , **1(1)** (1993), 5–11.

[3] B. C. Dhage, *Generalised metric spaces and mappings with fixed point*, Bulletin of Cal. Math. Soc., **84(4)** (1992), 329–336.

[4] S. Franklin, *Spaces in which sequences suffice*, Fundam. Math., **57(1)** (1965), 107–115.

[5] S. Gähler, *2-metrische räume und ihre topologische struktur*, Math. Nachr., **26(1-4)** (1963), 115–148.

[6] S. Gähler, *Lineare 2-normierte räume*, Math. Nachr., **28(1-2)** (1964), 1–43.

[7] S. Gähler, *Zur geometric 2-metriche räume*, Rev. Roum. Sci. Tech., Sér. Méc. Appl., **40** (1966), 664–669.

[8] S. Gähler, *Untersuchungen über verallgemeinerte m-metrische räume. I*, Math. Nachr., **40(1-3)** (1969),165–189.

[9] S. Gähler, *Untersuchungen über verallgemeinerte m-metrische räume. II*, Math. Nachr., **40(4-6)** (1969), 229–264.

[10] S. Gähler, *Untersuchungen über verallgemeinerte m-metrische räume. III*, Math. Nachr., **41(1-3)** (1969), 23–36.

[11] K. A. Khan, *Generalized normed spaces and fixed point theorems*, J. Math. Comput. Sci., **13** (2014), 157–167.

[12] A. Kundu, T. Bag, Sk. Nazmul, *A new generalization of normed linear space*, Topology Appl., **256** (2019),159–176.

[13] Z. Mustafa, H. Obiedat, F. Awawdeh, *Some fixed point theorem for mapping on complete G-metric spaces*, Fixed Point Theory Algorithms Sci. Eng., (2008).

[14] Z. Mustafa, V. Parvaneh, M. Abbas, J. R. Roshan, *Some coincidence point results for generalized (ψ, φ)-weakly contractive mappings in ordered G-metric spaces*, Fixed Point Theory Appl., **2013** (2013), 1–23.

[15] Z. Mustafa, V. Parvaneh, J. R. Roshan, Z. Kadelburg, *b_2-Metric spaces and some fixed point theorems*, Fixed Point Theory Appl., **2014(1)** (2014), 1–23.

[16] Z. Mustafa, W. Shatanawi, M. Bataineh, A. Volodin, *Existence of fixed point results in G-metric spaces*, Int. J. Math. Math. Sci., (2009).

[17] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7(2)** (2006), 289–297.

[18] K. P. R. Rao, S. V. R. Naidu, N. S. Rao, *On the concepts of balls in a D-metric space*, Int. J. Math. Math. Sci., **2005(1)** (2005), 133–141.

[19] S. Sedghi, I. Altun, N. Shobe, M. A. Salahshour, *Some properties of S-metric spaces and fixed point results*, Kyungpook Math. J., **54(1)** (2014), 113–122.

[20] S. Sedghi, N. Shobe, A. Aliouche, *A generalization of fixed point theorems in S-metric spaces*, Mat. Vesn., **64(249)** (2012), 258–266.

[21] S. Sedghi, N. Shobkolaei, J. R. Roshan, W. Shatanawi, *Coupled fixed point theorems in G_b-metric spaces*, Mat. Vesn., **66(2)** (2014), 190–201.

(received 04.07.2023; in revised form 12.03.2024; available online 03.09.2024)

Department of Basic Sciences and Humanities, Ramkrishna Mahato Government Engineering College, West Bengal, India
E-mail: anirbankundu92@gmail.com
ORCID iD: https://orcid.org/0000-0003-2537-7763

Department of Mathematics, Siksha-Bhavana, Visva-Bharati, West-Bengal, India
E-mail: tarapadavb@gmail.com
ORCID iD: https://orcid.org/0000-0002-8834-7097

Department of Mathematics, Kazi Nazrul University, Asansol, West-Bengal, India
E-mail: sk.nazmul_math@yahoo.in
ORCID iD: https://orcid.org/0000-0001-6674-1459