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DIVISIBLE LINEARLY ORDERED TOPOLOGICAL SPACES

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Abstract. We prove that a ccc linearly ordered topological space is metrizable if and only if it is divisible.

Let X be a topological space and A a subset of X. We will say that a family $\mathcal{D}_A$ of subsets of X is a *divisor for* A if for every $x \in A$ and every $y \in X \setminus A$ there exists $D \in \mathcal{D}_A$ such that $x \in D$ and $y \notin D$ [1]. If all members of $\mathcal{D}_A$ are closed (open, compact, ...) in X, then we say that $\mathcal{D}_A$ is a closed (open, compact, ...) divisor for A. In [1], A. Arhangel'skii defined a space X to be *divisible* if for every $A \subset X$ there is a countable closed divisor for A. The *divisibility degree* $dvs(X)$ of a space X is defined to be the smallest cardinal τ such that for every $A \subset X$ there exists a closed divisor for A having cardinality $\leqslant \tau$ [5], [6].

A family $\mathcal{U}$ of open subsets of a space X is called a *pseudobase* for X if for every $x \in X$ we have $\{x\} = \bigcap\{\, U \in \mathcal{U} \mid x \in U\,\}$. The *pseudoweight* of X, denoted by $pw(X)$, is defined by $pw(X) = \omega \cdot \min\{\, |\mathcal{U}| : \mathcal{U} \text{ is pseudobase for } X\,\}$.

Obviuosly, $dvs(X) \leqslant pw(X)$.

We use the usual topological terminology and notation following [2]; for definitions and results on cardinal functions we refer to [4]. w, pw, L, c, ψ denote the weight, pseudoweight, Lindelöf number, cellularity and pseudocharacter, respectively. All cardinals in this note are infinite.

Recall that a family γ of subsets of a set S is said to be *point separating* if for any $p, q \in S$, $p \neq q$, there is some $A \in \gamma$ such that $p \in A$ and $q \notin A$. We need the following known lemma:

1. Lemma. *If S is a set of cardinality $\leqslant 2^\gamma$, then there exists a point separating family γ of subsets of S having cardinality $\leqslant \gamma$.*

In [5] (see also [6], [7]), the following result was shown:

2. Theorem. *Every divisible compact Hausdorff space is metrizable.*

Here we prove that a ccc LOTS (= linearly ordered topological space) is divisible if and only if it is metrizable. In general, this result is not true for GO-spaces (= subspaces of LOTS's).

3. Theorem. *For any LOTS X we have $w(X) = c(X)dvs(X)$.*

Proof. Let $c(X)dvs(X) = \tau$. Since X is a LOTS then, as is well known [4], $|X| \leqslant c^{c(X)} \leqslant 2^{\tau}$. According to Lemma 1 there exists a point separating family $\{ S_\alpha \mid \alpha \in \tau \}$ of subsets of X. For every $\alpha \in \tau$ choose a closed divisor $\mathcal{D}_\alpha$ for S_α with $|\mathcal{D}_\alpha| \leqslant \tau$ and put $\mathcal{D} = \bigcup \{ \mathcal{D}_\alpha \mid \alpha \in \tau \}$. Then, $|\mathcal{D}| \leqslant \tau$ and $\mathcal{D}$ is a point separating family of closed subsets of X. Therefore, family $\mathcal{B} = \{ X \setminus D \mid D \in \mathcal{D} \}$ is a pseudobase for X of cardinality $\leqslant \tau$, i.e. $pw(X) \leqslant \tau$. By a result of K. P. Hart [3] (concerning LOTS's) we have $w(X) = c(X)pw(X) \leqslant \tau$. The oposite inequality $c(X)dvs(X) \leqslant w(X)$ is always true and the theorem is proved. ∎

4. Corollary. *A ccc LOTS X is divisible if and only if it is a separable metrizable space.*

Using this result we can once again get one known fact:

5. Example. The lexicographically ordered unit square is not a ccc space. Otherwise, this would mean that it is metrizable; but it is known that this is not true.

6. Remark. (1) The previous result is not valid in general for GO-spaces. The Sorgenfrey line S is a divisible (since $pw(X) \leqslant \omega$) ccc space, but S is not metrizable (even it is not developable).

(2) It is well known [2], [4] that every ccc LOTS is Lindelöf. So, it is natural to ask whether the conclusions of Corollary 4 can be extended to the class of Lindelöf LOTS's. It is not possible. In [3] there is an example of a LOTS X with $pw(X) = L(X) = \omega$ which is not metrizable. Of course, this space is divisible, because of $pw(X) = \omega$.

However, we have the following result.

7. Theorem. *If for every subset A of a LOTS X there exists a countable divisor consisting of closed Lindelöf G_δ-sets, then X is metrizable.*

Proof. First of all we prove that X is a Lindelöf space. Let X be any element in X. Let $\mathcal{D}_X = \{ L_i \mid i \in \omega \}$ be a countable divisor for $X \setminus \{x\}$ consisting of closed Lindelöf G_δ-sets. By the definition of a divisor we have $X \setminus \{x\} = \bigcup \{ L_i \mid i \in \omega \}$, so that $X \setminus \{x\}$ is a Lindelöf space. Thus X is also Lindelöf. It is known that every Lindelöf T_1-space in which for every $A \subset X$ there exists a countable divisor consisting of closed G_δ-sets has a G_δ-diagonal [1]. On the other hand, by a result of D. Lutzer [2], every LOTS with a G_δ-diagonal is metrizable. ∎

REFERENCES

[**1**] A. V. Arhangel'skii, *Some problems and lines of investigation in general topology*, Comment. Math. Univ. Carolinae **29** (1988), 611–629.

[**2**] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[**3**] K. P. Hart, *On the weight and pseudoweight of linearly ordered topological spaces*, Proc. Amer. Math. Soc. **82** (1981), 501–502.

[**4**] I. Juhász, *Cardinal functions in topology*, Mathematical Centre Tracts **34**, Amsterdam, 1971.

[**5**] Lj. Kočinac, *Compact divisible spaces are metrizable*, Abstracts Amer. Math. Soc. (1992).

[**6**] Lj. Kočinac, *The pseudoweight and splittability of a topological space*, Zbornik rad. Fil. fak. (Niš), Ser. Mat. **6** (1992), 191–197.

[**7**] Lj. Kočinac, *Clevability and divisibility of topological spaces*, atti, Accademia Pelor, dei Pericolanti **70** (1992) (to appear).

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