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A NOTE ON A SUPPORT OF A LINEAR MAPPING

Lj. Čukić

Abstract. In this note a notion of the support of a linear mapping from $C_b(T)$ into a locally convex space is introduced. Some of its properties are established.

Introduction

If E is a locally convex space and $P \subset E'$ is a weakly-$*$-bounded set, then P is equicontinuous iff the linear mapping f from E into the Banach space $C_b(P)$, defined by $f(e)(p) = p(e)$ $(p \in P,\ e \in E)$ is continuous. For the case $E = (C_b(T), \beta_t)$, as we will see, some information concerning the continuity of the mapping f is provided by its support.

Preliminaries

All topological spaces considered here are assumed to be completely regular Hausdorff. If T is such a spce, then $C_b(T)$ (resp. $C(T)$) denotes the space of bounded (resp. all) real-valued continuous functions on T. βT is the Stone-Čech compactification of T. For each $x \in C_b(T)$ its continuous extension to βT is denoted by x^β. If $x \in C(\beta T)$ and if $A \subset \beta T$, then $x|A$ denotes the restriction of x to A. $\mathrm{cl}_X A$ is the closure of $A \subset X$.

We denote by $\| \ \|$ supremum norm on $C_b(T)$, and by B the unit ball $\{ x \in C_b(T) : \|x\| \leqslant 1 \}$. $M(T)$ is the Banach space dual to $(C_b(T), \| \ \|)$. If $H \subset C_b(T)$ (or if $H \subset M(T)$), then H^+ denotes the set $\{ h \in H : h \geqslant 0 \}$. For such H, if $h \in H$, then $h^+ = \sup\{h, 0\}$, $h^- = \sup\{-h, 0\}$, $|h| = h^+ + h^-$.

Let t_{co} be the compact-open topology on $C_b(T)$, i.e. t_{co} is the locally convex topology on $C_b(T)$ defined by the family of seminorms $p_K(x) = \sup\{ |x(t)| : t \in K \}$, K runs through the compact subsets of T. Then, the strict topology β_t on $C_b(T)$ is the finest locally convex topology on $C_b(T)$ coinciding with t_{co} on the unit ball B ([2],[6]). From definition of β_t immediately follows that if f is a linear mapping from $C_b(T)$ into an LCS (a locally convex Hausdorff space) then f is β_t-continuous iff its restriction $f|B$ is t_{co} - continuous. $M_t(T)$ denotes the continuous dual of $(C_b(T), \beta_t)$.

Results

A well-known theorem of Nachbin (see [5], III.1.2) says that if $F \subset C(T)$ is absolutely convex and if $\varepsilon B \subset F$ for some $\varepsilon > 0$, then there is a minimal compact set $K \subset \beta T$ with the property: if $x \in C(T)$ and if $x^\beta | K = 0$, then $x \in F$. We prove the following variant of Nachbin's theorm.

Theorem 1. *If $F \neq \{0\}$ is a non-empty norm-closed absolutely convex subset of $C_b(T)$, then there is a minimal compact set $S(F) \subset \beta T$ with the property: if $x \in C_b(T)$ and if $x^\beta | S(F) = 0$, then $x \in F$.*

Proof. Let $\mathcal{L} = \{\, L \subset \beta T \,:\, L$ is compact such that $(\forall x \in C_b(T)) x^\beta | L = 0 \Rightarrow x \in F \,\}$ and let $\mathcal{M}_K = \{\, x \in C_b(T) \,:\,$ there exists an open $G \supset K$ with $x^\beta | G = 0 \,\}$, for compact $K \subset \beta T$. Then: (1) $L \in \mathcal{L}$ iff $\mathcal{M}_L \subset F$; (2) if $L_1, L_2 \in \mathcal{L}$, then $L_1 \cap L_2 \in \mathcal{L}$; (3) $S(F) = \bigcap \{\, L \,:\, L \in \mathcal{L} \,\}$. Proofs of (2) and (3) are the same as in [5], pp. 63–64. One half of (1) is trivial. To obtain the other half, suppose that $\mathcal{M}_L \subset F$ and $x \in C_b(T)$, $x^\beta | L = 0$. Let $y_n(t) = x^\beta(t)$ if $|x^\beta(t)| < 1/n$ and $y_n(t) = x^\beta(t)/(n|x^\beta(t)|)$ if $|x^\beta(t)| \geqslant 1/n$. Then $y_n \in C(\beta T)$ and $(x^\beta - y_n)|G_n = 0$, for $G_n = \{\, t \in \beta T \,:\, |x^\beta(t)| < 1/n \,\}$. From $L \subset G_n$ it follows that $(x^\beta - y_n)|T \in F$ for each $n = 1, 2, \dots$. Then $x \in F$, because F is closed and $\|y_n|T\| \leqslant 1/n$. ∎

Remark 2. If F is as in theorem 1 and if F is norm-bounded, then $S(F) = \beta T$. In fact, if $t \in \beta T \setminus S(F)$, then there is $x \in C_b(T)$ with $x^\beta(t) = 1$, $x^\beta | S(F) = 0$. Hence $nx \in F$, because $nx^\beta | S(F) = 0$ $(n = 1, 2, \dots)$, i.e. F is not norm-bounded.

Definition 3. Let $f \neq 0$ be a norm-continuous linear mapping from $C_b(T)$ into an LCS E. The big support of f is $\operatorname{bsupp} f = S(f^{-1}(0))$ and the support of f is $\operatorname{supp} f = \operatorname{bsupp} f \cap T$.

Remark 4. If f is a norm-continuous linear functional on $C_b(T)$, then f can be identified, via Alexandroff representation theorem ([6], 5.1) with the unique Baire measure μ on the minimal algebra which contains all zero sets from T. It is not difficult to see that $\operatorname{supp} f$ and $\operatorname{supp} \mu$ coincide.

In the light of the preceeding remark, next result is not new, but we give a proof which is independent from the measure theory.

Proposition 5. *Let $f \in M^+(T)$ and $f \neq 0$. Then:*

(a) If $x \in C_b^+(T)$ and $f(x) = 0$, then $x^\beta | \operatorname{bsupp} f = 0$.

(b) The space $\operatorname{bsupp} f$ with the induced topology satisfies the countable chain condition.

Proof. (a) Let $x^\beta(s) > 0$ for some $s \in \operatorname{bsupp} f$. Then there exist an open set $G \subset \beta T$ and $r > 0$ with $x^\beta(t) > r$ for all $t \in G$. We will prove that $\operatorname{bsupp} f$ is contained in $\beta T \setminus G$, which is impossible because $s \in \operatorname{bsupp} f \cap G$. Let $y \in C_b(T)$, $y^\beta | \beta T \setminus G = 0$ and $\|y\| < k$. From $x^\beta(t) > r(y^\pm)^\beta(t)/k$ for all $t \in \beta T$ and from non-negativity of f it follows that $f(y^\pm) = 0$. Then $f(y) = f(y^+) - f(y^-) = 0$. Hence $\operatorname{bsupp} f \subset \beta T \setminus G$, by the minimality of bsupp.

(b) Let $S = \operatorname{bsupp} f$ and let the functional g_0 on $C_b^+(S)$ be defined by $g_0(x) = f(\bar{x})$, where $\bar{x}$ is any non-negative continuous extension of $x \in C_b^+(S)$ on βT. The functional g_0 is well-defined because each two such extensions coincide on S. It is trivial to see that g_0 is a non-negative additive functional, and by [1, Chap.II, §2, Prop.2] there is a non-negative linear functional g on $C_b(S)$ that extends g_0. By [4, V.5.5], $g \in M^+(S)$.

Let $\{ G_\alpha : \alpha \in A \}$ be a family of non-empty pairwise disjoint open subsets of S and let $t_\alpha \in G_\alpha$. Then there are $x_\alpha \in C_b^+(S)$, $x_\alpha \leqslant 1$, such that $x_\alpha(t_\alpha) = 1$ and $x_\alpha | S \setminus G_\alpha = 0$. From $0 \leqslant \sum_\Phi x_\alpha \leqslant 1$ on S it follows that $0 \leqslant \sum_\Phi g(x_\alpha) \leqslant g(1)$ for all finite $\Phi \subset A$. Then the set $\{\alpha \in A : g(x_\alpha) \geqslant g(1)/n \}$ is finite for each $n \in \mathbf{N}$. Countability of A then follows from the inequality $g(x_\alpha) > 0$ (by (a)). ∎

Theorem 6. *Let E be a metrizable LCS, let (U_n) be its neighborhood basis of origin consisting of absolutely convex sets with $2U_{n+1} \subset U_n$, and let $f \neq 0$ be a norm-continuous linear mapping from $C_b(T)$ into E. Then f is β_t-continuous if and only if there are compact sets $L_n \subset T$ $(n \in \mathbf{N})$ with the property that $f(x) \in U_n$, whenever $x \in B^+$ and $x|L_n = 0$. Moreover, L_n's may be chosen such that* $\operatorname{supp} f = \operatorname{cl}_T(\bigcup_{n=1}^\infty L_n)$.

Proof. $\Longrightarrow$ The restriction $f|B$ is t_{co}-continuous. Then there are an increasing sequence of compact sets $K_n \subset T$ and a decreasing sequence ε_n of positive numbers with the property: if $x \in B$ and $p_{K_n}(x) < \varepsilon_n$, then $f(x) \in U_n$. We will first prove that $\bigcup_{n=1}^\infty K_n \cap \operatorname{bsupp} f \neq \emptyset$. Suppose the contrary. Then, there are $x_n \in B^+$ such that $x_n | K_n = 0$, $x_n | \operatorname{bsupp} f = 1$. There is $u \in B$ such that $f(u) \neq 0$. From $(ux_n)^\beta | K_n = 0$, $(ux_n)^\beta | \operatorname{bsupp} f = u^\beta | \operatorname{bsupp} f$ it follows that $f(u) = f(ux_n) \in U_n$ for all n, which is in contradiction with $f(u) \neq 0$.

Hence, there is $k \in \mathbf{N}$ such that $K_n \cap \operatorname{bsupp} f$ is non-empty for all $n \geqslant k$. Let $L_n = K_{n+k} \cap \operatorname{bsupp} f$, $\delta_n = \varepsilon_{n+k}$ and let $x \in B^+$, $x|L_n = 0$. If $K_{n+k} \subset G_n = \{ t \in \beta T : x^\beta(t) < \delta_n \}$ then $f(x) \in U_{n+k} \subset U_n$. If $K_{n+k} \not\subset G_n$, then from $L_n \subset G_n$ it follows that there is $y \in B^+$ with $y^\beta | K_{n+k} \cap (T \setminus G_n) = 0$, $y^\beta | \operatorname{bsupp} f = 1$. Since $K_{n+k} = (L_n \cup (K_{n+k} \setminus (T \setminus G_n))) \cup (K_{n+k} \cap (G_n \setminus L_n))$, then $p_{K_{n+k}}(xy) < \delta_n$. From this and from the fact that x^β and $(xy)^\beta$ coincide on $\operatorname{bsupp} f$ it follows that $f(x) = f(xy) \in U_{n+k} \subset U_n$.

For the equality $\operatorname{supp} f = \operatorname{cl}_T(\bigcup_{n=1}^\infty L_n)$, only inclusion $\operatorname{bsupp} f \subset \operatorname{cl}_{\beta T}(\bigcup_{n=1}^\infty L_n)$ needs a proof. If $z \in C_b(T)$ and $z^\beta | \operatorname{cl}_{\beta T}(\bigcup_{n=1}^\infty L_n) = 0$, then $(z^\pm / \|z\|) \in B^+$ and $(z^\pm / \|z\|) | L_n = 0$. It follows that $f(z^\pm) \in \|z\| U_n$ for all n, i.e. $f(z) = 0$. By the minimality of $\operatorname{bsupp} f$, the proof is finished.

$\Longleftarrow$ Since f is norm-continuous, we may choose positive numbers $a_n < 1$ so that $4f(a_n B) \subset U_{n+1}$ for each n. We will show that $f(V_n \cap B) \subset U_n$, where V_n is the set $\{ x \in C_b(T) : p_{L_{n+2}}(x) < a_n \}$. Let $x \in V_n \cap B$ and let

$$y^+(t) = \begin{cases} x^+(t), & \text{if } x^+(t) < a_n, \\ a_n, & \text{if } x^+(t) \geqslant a_n, \end{cases} \qquad y^-(t) = \begin{cases} x^-(t), & \text{if } x^-(t) < a_n, \\ a_n, & \text{if } x^-(t) \geqslant a_n. \end{cases}$$

Then $x^\pm - y^\pm \in B^+$, $y^\pm \in a_n B$, $(x^\pm - y^\pm) | L_{n+2} = 0$, and so $f(x^\pm - y^\pm) \in U_{n+2}$ and $4f(y^\pm) \in U_{n+1}$. From this it follows that $f(x) = f(x^+) - f(x^-) = (f(x^+ -$

$y^+) + f(y^+)) - (f(x^- - y^-) + f(y^-)) \in 2U_{n+2} + \frac{1}{2}U_{n+1} \subset U_n$, which completes the proof of the theorem. ■

Remark 7. If E is a non-metrizable LCS, then $\operatorname{supp} f$ need not be the closure of a σ-compact subset of T, as the following example shows. Let T be the discrete space, $\operatorname{card} T = c$. Then T is a realcomplete ([3,11.D.(a)]) metrizable space. By [5,III.3.5 and III.4.3] $E = (C(T), t_{co})$ is a bornological barrellled complete LCS. The inclusion mapping i from $C_b(T)$ into E is β_t-continuous and from remark 2 it follows that $\operatorname{supp} i = T$. Each compact subset of T is finite, hence $\operatorname{cl}_T(\bigcup_{n=1}^\infty L_n) = \bigcup_{n=1}^\infty L_n \neq T$ for all compact L_n's.

Remark 8. In the proof of theorem 6 we showed also that $\operatorname{cl}_T \bigcup_{n=1}^\infty L_n$ is dense in $\operatorname{bsupp} f$. Hence, $\operatorname{supp} f$ is dense in $\operatorname{bsupp} f$.

The next lemma is well-known and we omit the proof.

Lemma 9. *Let $f_n \in M_t^+(T)$, $\|f_n\| \leqslant 1$ and let $f = \sum_{n=1}^\infty 2^{-n} f_n$. Then $f \in M_t^+(T)$, $\|f\| \leqslant 1$ and* $\operatorname{supp} f = \operatorname{cl}_T(\bigcup_{n=1}^\infty \operatorname{supp} f_n)$.

Theorem 10. *Let $f \neq 0$ be a weakly continuous linear mapping from $(C_b(T), \beta_t)$ into an LCS E. Then:*

(a) If $F \subset E'$ is weakly-$*$-dense in E', then $\operatorname{supp} f = \operatorname{cl}_T(\bigcup_{w\in F} \operatorname{supp}(wf))$.

(b) $\operatorname{supp} f$ *is dense in* $\operatorname{bsupp} f$.

(c) If E' is weakly-$*$-separable, then there is $\mu \in M_t^+(T)$, $\|\mu\| \leqslant 1$ such that $\operatorname{supp} f = \operatorname{supp} \mu$.

(d) If E' is weakly-$*$-separable, then $\operatorname{supp} f$ *satisfies the countable chain condition.*

Proof. (a) From $f^{-1}(0) \subset (wf)^{-1}(0)$ and the theorem 1 it follows that $\operatorname{bsupp}(wf) \subset \operatorname{bsupp} f$ for each $w \in F$. On the other hand, if $x^\beta | \operatorname{cl}_{\beta T}(\bigcup_{w\in F} \operatorname{supp}(wf)) = 0$, then by the remark 8, $x^\beta | \operatorname{bsupp}(wf) = 0$. This gives that $wf(x) = 0$ for all $w \in F$. Hence $f(x) = 0$. From the theorem 1 it follows that $\operatorname{bsupp} f \subset \operatorname{cl}_{\beta T}(\bigcup_{w\in F} \operatorname{supp}(wf))$.

(b) Immediately follows from (a).

(c) Let $\{ w_n : n \in \mathbf{N} \}$ be weakly-$*$-dense in E'. Then $\operatorname{supp} f = \operatorname{cl}_T(\bigcup_{n=1}^\infty \operatorname{supp}(w_n f))$, by (a). If $\mu = \sum_{n\in N_1} 2^{-n}(|w_n f| / \|w_n f\|)$, where $N_1 = \{ n : w_n f \neq 0 \}$, then $\mu \in M_t^+(T)$ and $\operatorname{supp} f = \operatorname{supp} \mu$, by the lemma 9.

(d) From (b) and (c) it follows that $\operatorname{supp} f$ is dense in $\operatorname{bsupp} \mu$. Then, by the proposition 5, from [3, 2.J.(d)] it follows that $\operatorname{supp} f$ satisfies the countable chain condition. ■

Remark 11. Assertions in (c), (d) are not true if we omit the separability condition, even if E is a Banach space. For example, let T be the compact space $\beta\mathbf{N} \setminus \mathbf{N}$ and let f be the identity mapping on $(C(T), \|\ \|)$. Then $\operatorname{supp} f = T$, but T does not satisfy the countable chain condition [3, 3.6.Example 2].

Applications of our results will be given in a subsequent paper.

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Department of Mathematics, Faculty of Civil Engineering, Bul. Revolucije 73, 11000 Beograd, Yugoslavia