МАТЕМАТИЧКИ ВЕСНИК
47 (1995), 11–18

UDC 515.123.2
оригинални научни рад
research paper

SOME PROPERTIES OF SEMI-CONTINUOUS FUNCTIONS AND QUASI-UNIFORM SPACES

J. Ferrer, V. Gregori and I. L. Reilly

Abstract. This paper approaches the problem of characterizing the topological spaces that admit a unique compatible quasi-uniform structure by means of semi-continuous functions. The desired result is achieved by relating properties of the semi-continuous quasi-uniformity with a new characterization of hereditarily compact spaces via lower semi-continuous functions.

1. Introduction

In the following we establish the basic definitions, general results and terminology to be used throughout this paper; they are found mostly in [2].

By a quasi-uniform space $(X,\mathcal{U})$ we mean the natural analog of a uniform space obtained by dropping the symmetry axiom. For each $x \in X$ and $U \in \mathcal{U}$, $U(x) = \{ y \in X : (x,y) \in U \}$ defines a basic neighborhood of x for a topology in X which is the topology induced by $\mathcal{U}$. Considering the collection $\mathcal{U}^{-1}$ of the sets $U^{-1} = \{ (x,y) : (y,x) \in U \}$, $U \in \mathcal{U}$, we obtain the conjugate quasi uniform space $(X,\mathcal{U}^{-1})$. The collection of sets of the form $U \cap U^{-1}$, $U \in \mathcal{U}$, is a subbase for a uniform structure $(X,\mathcal{U}^*)$, the uniform space induced by $(X,\mathcal{U})$.

A quasi-uniform space $(X,\mathcal{U})$ is precompact (totally bounded) provided that for each $U \in \mathcal{U}$ there is a finite subset F of X (finite cover $\mathcal{C}$ of X) such that $U(F) = X$ ($A \times A$ is contained in U, for each $A \in \mathcal{C}$). Although total boundedness implies precompactness, the converse is not true for quasi-uniform spaces in general.

In any topological space X, for each open set G, the set $S(G) = \{ (x,y) : x \notin G$, or $x,y \in G \}$ defines a subbasic entourage of a compatible transitive totally bounded quasi-uniformity $\mathcal{P}$ usually referred to as Pervin's quasi-uniformity [9], which is the finest of all totally bounded compatible quasi-uniformities in X [2, Prop. 2.2].

By dropping the symmetry axiom of a pseudometric (metric) one obtains a quasi-pseudometric (quasi-metric). The euclidean uniform structure $\mathcal{E}$ of the real

---

AMS Subject Classification: $54E35, 54E55$

Keywords: Semi-continuous function, quasi-uniform space, hereditarily compact space

The first two authors have been partially supported by a grant from the DGICYT, No.PB89-0611, and the third author by a grant from the University of Auckland Research Committee.

line $\mathbf{R}$ may be decomposed into two conjugate quasi-pseudometric spaces $(\mathbf{R}, \mathcal{Q})$, $(\mathbf{R}, \mathcal{Q}^{-1})$, where $\mathcal{Q}$ denotes the quasi-uniformity induced by the quasi-pseudometric $d(x,y) = \max\,(x-y, 0)$, i.e., the sets $V_r = \{\,(x,y) : x-y < r\,\}$, $r > 0$, form a base for $\mathcal{Q}$ and $\mathcal{Q}^* = \mathcal{E}$.

For a function between two quasi-uniform spaces, quasi-uniform continuity is defined in the analogous way to the uniform case.

The symbols $LSC(X)$, $USC(X)$, $LSCB(X)$, $USCB(X)$, $LSCBB(X)$, $USCBA(X)$ will indicate the collections of all real valued functions which are lower semi-continuous, upper semi-continuous, lower semi-continuous bounded, upper semi-continuous bounded, lower semi-continuous bounded below and upper semi-continuous bounded above, respectively.

Given a quasi-uniform space $(X,\mathcal{U})$, the symbols $Q(\mathcal{U})$, $Q^{-1}(\mathcal{U})$, $E(\mathcal{U})$, $[QB(\mathcal{U})$, $Q^{-1}B(\mathcal{U})$, $EB(\mathcal{U})]$ will stand for the collections of all real valued functons in X which are quasi-uniformly continuous [and bounded] from $(X,\mathcal{U})$ to $(\mathbf{R}, \mathcal{Q})$, $(\mathbf{R}, \mathcal{Q}^{-1})$ and $(\mathbf{R}, \mathcal{E})$, respectively.

It is well known that the ring of continuous real valued functions $C(X)$ on a completely regular space determines a compatible uniformity $\mathcal{C}$, which is the coarsest for which every element of $C(X)$ is uniformly continuous. In the same way, given any topological space X, for each $f \in LSC(X)$ and $r > 0$, the set $U(f,r) = \{\,(x,y) : f(x) - f(y) < r\,\}$ defines a subbasic entourage for a compatible quasi-uniformity $\mathcal{SC}$ which is the coarsest of all compatible quasi-uniformities $\mathcal{U}$ for which $LSC(X)$ is contained in $Q(\mathcal{U})$, see [2, Prop. 2.10].

The purpose of this paper is to relate semi-continuity to quasi-uniform continuity by means of the quasi-uniformity $\mathcal{SC}$. This relationship will turn out to be useful as it will provide a characterization of those topological spaces that admit a unique compatible quasi-uniform structure. Lindgren [8] has shown that a topological space admits a unique compatible quasi-uniformity if and only if every compatible quasi-uniformity is totally bounded.

No separation properties are assumed for the spaces used throughout this paper unless explicitly stated.

2. Semi-continuity and quasi-uniform continuity

In this section we compare the notions of semi-continuous and quasi-uniformly continuous functions with respect to the structures $\mathcal{Q}$, $\mathcal{Q}^{-1}$ and $\mathcal{E}$, and obtain the interesting equality $E(\mathcal{P}) = C^*(X)$, i.e., the bounded continuous functions in X coincide with the functions that are quasi-uniformly continuous with respect to Pervin's quasi-uniformity. We also show that the quasi-uniform space $(X, \mathcal{SC})$ is totally bounded if and only if every semi-continuous function in X is bounded.

2.1. Proposition. *If $(X,\mathcal{U})$ is a precompact quasi-uniform space, then the following inclusion relations hold*

$$Q(\mathcal{U}) \subset LSCBB(X), \quad Q^{-1}(\mathcal{U}) \subset USCBA(X), \quad E(\mathcal{U}) \subset C^*(X).$$

Proof. We show only the first inclusion since the second is analogous and the third one follows from the first two and the fact that $E(\mathcal{U}) = Q(\mathcal{U}) \cap Q^{-1}(\mathcal{U})$.

Let $f \in Q(\mathcal{U})$. Since f is continuous from X to $(\mathbf{R}, \mathcal{Q})$, we have $f \in LSC(X)$. By quasi-uniform continuity, there is $U \in \mathcal{U}$ such that $f(x) - f(y) < 1$, for all $(x, y) \in U$. Since $(X, \mathcal{U})$ is precompact, there is a finite subset F of X such that $X = U(F)$. Let $r = \min\{\, f(x) - 1 : x \in F\,\}$, then $f(x) > r$, for all x. Thus, $f \in LSCBB(X)$. ■

2.2 Proposition. *For any topological space X the following inclusion relations hold*

$$LSCB(X) \subset Q(\mathcal{P}), \quad USCB(X) \subset Q^{-1}(\mathcal{P}).$$

Proof. Again, we prove only the first inclusion. Let f be in $LSCB(X)$. For each $r > 0$, since f is bounded, there is a finite subset F of $\mathbf{R}$ such that $f(X)$ is contained in the finite union of open intervals $\bigcup\{\,(t - r/2, t + r/2) : t \in F\,\}$.

For each $t \in F$, set $G(t) = \{\, x \in X : f(x) > t - r/2\,\}$. Then $G(t)$ is an open subset of X and the entourage $U = \bigcap\{\, S(G(t)) : t \in F\,\}$ is in $\mathcal{P}$.

Now, for $(x, y) \in U$, let $t \in F$ be such that $t - r/2 < f(x) < t + r/2$.

Then $x \in G(t)$ and $y \in G(t)$. Thus $f(x) - f(y) = f(x) - t + t - f(y) < r$, and $f \in Q(\mathcal{P})$. ■

2.3 Corollary. *In any topological space X the following relations hold*

$$LSCB(X) = QB(\mathcal{P}) \subset Q(\mathcal{P}) \subset LSCBB(X),$$
$$USCB(X) = Q^{-1}B(\mathcal{P}) \subset Q^{-1}(\mathcal{P}) \subset USCBA(X),$$
$$EB(\mathcal{P}) = E(\mathcal{P}) = C^*(X).$$

2.4 Proposition. *In a topological space X, the semi-continuous quasi-uniformity $\mathcal{SC}$ is totally bounded if and only if every semi-continuous function is bounded.*

Proof. For the necessity part, it suffices to show that $LSC(X) = LSCB(X)$. Let $f \in LSC(X)$. There is a finite cover $A_1, A_2, \ldots, A_n$ of X such that $A_i \times A_i \subset U(f, 1)$, $1 \leq i \leq n$. For each $i = 1, 2, \ldots, n$, choose $x_i \in A_i$. Then,

$$A_i \subset U(f, 1)(x_i) \cap U(f, 1)^{-1}(x_i) = f^{-1}(f(x_i) - 1, f(x_i) + 1).$$

Thus, $f(A_i)$ is bounded for $1 \leq i \leq n$, and so is $f(X)$.

Conversely, from Proposition 2.2, $LSC(X) = LSCB(X) \subset Q(\mathcal{P})$. But the construction of $\mathcal{SC}$ implies that it is equal to $\mathcal{P}$. Since $\mathcal{P}$ is totally bounded, so is $\mathcal{SC}$. ■

3. Semi-continuous functions and unique quasi-uniformity spaces

The problem of finding a purely topological characterization of the spaces that admit only one compatible quasi-uniformity was first posed by Fletcher [1] in 1969. A few years ago, Künzi [6] came up with the desired characterization by showing that a topological space admits a unique quasi-uniformity if and only if it is hereditarily compact (every subset is compact) and it has no strictly decreasing sequence of open sets with open intersection.

Our aim here is to achieve a similar result to that of Künzi in terms of lower semi-continuous functions.

Hereditarily compact spaces were studied intensively by Stone [10]. He characterized them in several different ways, although not in terms of semi-continuous functions. Next, we give a characterization of hereditariy compact spaces using semi-continuous functions which will be needed later.

3.1. PROPOSITION. *A topological space X is hereditarily compact if and only if the range $f(X)$ of every lower semi-continuous function f is well ordered in* $\mathbf{R}$.

Proof. Let $f \in LSC(X)$. If $f(X)$ were not well ordered in $\mathbf{R}$, we could construct a strictly decreasing sequence $(f(x_n))$. For each n, let $G_n = \{ x \in X : f(x) > f(x_n) \}$. Then (G_n) is a strictly increasing sequence of open sets, which contradicts the fact that every subset is compact.

Conversely, if X were not hereditarily compact, we could find a strictly increasing sequence of open sets (G_n). For each n, let g_n be the characteristic function of G_n. The function

$$f = \sup_n \left(\frac{g_n}{n} \right)$$

is then lower semi-continuous, but $f(X)$ is not well ordered in $\mathbf{R}$, since it contains the sequence $1, 1/2, 1/3, \ldots$. ■

In the proof of next theorem we make use of Künzi's characterization of the spaces that admit a unique compatible quasi-unformity, [6].

3.2 THEOREM. *For a topological space X the following statements are equivalent:*

(i) *X admits a unique compatible quasi-uniformity.*

(ii) *For every $f \in LSC(X), f(X)$ is well ordered and compact.*

(iii) *For every $f \in LSC(X), f(X)$ is well ordered and closed.*

Proof. If X has a unique compatible quasi-uniformity, then it is shown in [6] that X is hereditarily compact. Let $f \in LSC(X)$. By Proposition 3.1, $f(X)$ is well ordered. Also, since the quasi-uniformity $\mathcal{SC}$ is totally bounded, $f(X)$ is bounded, by Proposition 2.4.

We show that $f(X)$ is closed. Let $(f(x_n))$ be a strictly increasing sequence convergent to $t \in \mathbf{R}$ (if t were the limit of a non-increasing sequence in $f(X)$, we would clearly have $t \in f(X)$). For each n, let $G_n = \{ x \in X : f(x) > f(x_n) \}$. Then

(G_n) is a strictly decreasing sequence of open sets whose intersection is $f^{-1}([t,\infty))$. If $t \notin f(X)$, we have

$$\bigcap_{n=1}^{\infty} G_n = f^{-1}((t,\infty))$$

which is open, thus contradicting Künzi's result. This shows that (i) implies (ii).

The implication (ii) implies (iii) being obvious, we show that (iii) implies (i). From Proposition 3.1, we have that X is hereditarily compact. Suppose (G_n) is a strictly decreasing sequence of open sets with open intersection. For each n, let g_n, g denote the characteristic functions of G_n, $\bigcap_{n=1}^{\infty} G_n$, respectively. Let $f = \sup(1-1/n)g_n$, $h = \max(f, 2g)$. The functions g_n, g, f and h are all in $LSC(X)$, but $h(X)$ is not closed since it contains the sequence $(1-1/n)$ and $1 \notin h(X)$. ■

4. The semi-continuous quasi-uniformity and Fletcher's conjecture

Fletcher [1] conjectured that a space with unique compatible quasi-uniformity should have a finite topology. Simple counterexamples to the conjecture can be found in [3] and [8]. Nevertheless, it is shown in [2] that the assertion remains valid for large classes such as the Hausdorff or regular spaces.

In this section we see that the semi-continuous quasi-uniformity $\mathcal{SC}$ suffices to extend the validity of Fletcher's conjecture to the classes of quasi-regular and R_1 spaces.

A topological space X is said to be quasi-regular whenever every non-empty open set contains the closure of some non-empty open set. This class of spaces is introduced in [4] for the study of Baire like properties. The next theorem provides a parallel result to the one stated in [2, Th.2.36].

4.1 Theorem. *In a quasi-regular space X the following statements are equivalent:*

(i) *X has a unique compatible quasi-uniformity.*

(ii) *The quasi-uniformity $\mathcal{SC}$ is totally bounded.*

(iii) *Every semi-continuous function is bounded.*

(iv) *X is hereditarily compact.*

(v) *The topology of X is finite.*

Proof. The implications (i) $\to$ (ii) and (ii) $\to$ (iii) are evident from previous comments and results.

We show that (iii) implies (iv). Suppose X is not hereditarily compact. Then there is a strictly increasing sequence (G_n) of open sets. Making use of the quasi-regularity of X, a sequence (H_n) of proper open sets may be constructed inductively such that

$$\mathrm{cl}H_1 \subset G_1, \quad \text{and} \quad \mathrm{cl}H_n \subset G_n - \bigcup_{i=1}^{n-1} \mathrm{cl}H_i, \qquad n \geq 2.$$

For each n, let h_n be the characteristic function of H_n. Then, the function $f = \sup(n \cdot h_n)$ is an unbounded element of $LSC(X)$.

Since the implication (v) $\rightarrow$ (i) holds in general [2, Th.2.36] we need only show that (iv) implies (v). For each open set G we assert the existence of a minimal proper clopen set contained in G. If not, we could define inductively a sequence (H_n) of proper open sets such that

$$H_1 \subset G, \quad \text{and} \quad \mathrm{cl}H_n \subset H_{n-1}, \qquad n \geq 2$$

where the inclusions are strict. Thus the set

$$A = \bigcup_{n-1}^{\infty} (H_n - \mathrm{cl}H_{n+1}) \quad \text{would not be compact.}$$

Now suppose the topology is infinite. We construct inductively a sequence (H_n) of minimal proper clopen sets such that, for each n, H_{n+1} is disjoint with $H_1 \cup H_2 \cup \cdots \cup H_n$.

We start out with any minimal proper clopen set H_1, and assume $H_1, \ldots, H_n$ are already defined. Obviously, the sets $H_1, H_2, \ldots, H_n$ do not cover X, otherwise their minimality would turn them into a base for the topology, contradicting our assumption that the topology is infinite. Thus, there is a minimal proper clopen set H_{n+1} contained in $X - (H_1 \cup H_2 \cup \cdots \cup H_n)$.

The existence of the sequence (H_n) contradicts the fact that X is hereditarily compact since

$$\bigcup_{n=1}^{\infty} H_n \quad \text{cannot be compact.} \quad \blacksquare$$

It is shown in [2, Th.3.22] that, for Hausdorff spaces, total boundedness of $\mathcal{SC}$ is equivalent to having a finite topology. We proceed to extend this result to the class of R_1 spaces. Recall that a topological space is R_1 whenever $\mathrm{cl}\{x\} \neq \mathrm{cl}\{y\}$ implies the existence of disjoint neighbourhoods of x and y.

4.2 Lemma. *Let X be an R_1 space and set $A = \{x \in X : \mathrm{cl}\{x\}$ is open $\}$. If $X - A$ is a non-empty clopen set, then there is a sequence (H_n) of pairwise disjoint proper open sets in $X - A$ and thus in X.*

Proof. Let $x \in X - A$, then $\mathrm{cl}\{x\} \subsetneqq X - A$. Thus, there is $y \in X - A$ such that $\mathrm{cl}\{x\} \neq \mathrm{cl}\{y\}$. Since X is R_1, there exist H_1, V_1 which are disjoint open neighbourhoods of x, y, respectively, contained in $X - A$. Since $\mathrm{cl}\, H_1$ is a proper subset of $X - A$, there is $p \in (X - A) - \mathrm{cl}\, H_1$, and $\mathrm{cl}\{p\} \subset (X - A) - \mathrm{cl}\, H_1$. Thus, there is $q \in (X - A) - \mathrm{cl}\, H_1$ with $\mathrm{cl}\{p\} \neq \mathrm{cl}\{q\}$. As before, there are H_2, V_2, disjoint open neighbourhoods of p, q, contained in $(X - A) - \mathrm{cl}\, H_1$, and $H_1 \cap H_2 = \phi$, such that $(\mathrm{cl}\, H_1 \cup \mathrm{cl}\, H_2)$ is a proper subset of $X - A$. Repeating this process inductively one obtains the sequence (H_n) of pairwise disjoint proper open sets. $\blacksquare$

4.3 Theorem. *In an R_1 space X, the quasi-uniformity $\mathcal{SC}$ is totally bounded if and only if the topology of X is finite.*

Proof. Sufficiency is certainly true in general. For the necessity part, consider the set A as in the previous lemma. Then $A \neq \phi$, otherwise the sequence (H_n) of pairwise disjoint proper open sets would provide a function $f = \sup(n \cdot g_n)$ (g_n being the characteristic function of H_n) which is lower semi-continuous and not bounded, contradicting Proposition 2.4.

Furthermore, the collection $\{ \operatorname{cl}\{x\} : x \in A \}$ must be finite. On the other hand, we could choose an infinite sequence $(\operatorname{cl}\{x_n\})$ of pairwise disjoint open sets, thus yielding a contradiction as before. Therefore, since $A = \bigcup \{ \operatorname{cl}\{x\} : x \in A \}$, it is a clopen set and so is $X - A$. By previous lemma, $X - A = \phi$. Thus $A = X$, and the finite collection $\{ \operatorname{cl}\{x\} : x \in A \}$ is a base for the open sets, thereby defining a finite topology. ■

Notice that, under the assumptions of the last theorem, the finite topology obtained is strongly zero-dimensional.

4.4 Corollary. *For regular (Hausdorff) spaces, $\mathcal{SC}$ is totally bounded if and only if the topology (the ground set) is finite.*

It is shown in [8] that a topological space X admits a unique compatible quasi-uniformity if and only if every compatible quasi-uniformity is totally bounded. We have just seen that, for R_1 spaces, total boundedness of $\mathcal{SC}$ suffices. However, if we remove the R_1 assumption, it is no longer true that $\mathcal{SC}$ totally bounded implies a unique quasi-uniformity. As we see in the following example, it does not even imply the weaker property of having a unique quasi-proximity.

4.5 Example. Let ω_1 denote the first uncountable ordinal. In the ordinal space $X = [1, \omega_1]$, consider the quasi-pseudometric:

$$d(x, y) = 0 \quad \text{if} \quad x \geq y, \qquad d(x, y) = 1 \quad \text{otherwise},$$

which generates the bitopological space (X, T, T'), where T and T' are the topologies induced by d and its conjugate $d'(x, y) = d(y, x)$.

It is easy to see that every T-semi-continuous function is bounded, and the same applies for T'. Thus, their semi-continuous quasi-uniformities $\mathcal{SC}$ and $\mathcal{SC}'$ are both totally bounded.

Nevertheless, from Theorem 1 of [5], (X, T) admits more than one compatible quasi-proximity, since the family of open sets $[1, x]$, $x < \omega_1$, is a base for T which is closed under finite intersections and unions. Also, (X, T') admits more than one quasi-uniformity, since the FINE quasi-uniformity formed by all supersets of $U = \{ (x, y) : x \leq y \}$ is not totally bounded.

REFERENCES

[**1**] Fletcher, P., *Finite topological spaces and quasi-uniform structures*, Canad. Math. Bul. **12** (1969), 771–775

[**2**] Fletcher, P. and Lindgren, W.F., *Quasi-uniform Spaces*, Marcel Dekker, New York 1982

[**3**] Gregori, V. and Ferrer, J., *Some classes of topological spaces with a unique quasi-uniformity*, Canad. Math. Bul. **29** (1986), 446–449

[**4**] Haworth, R.C.and McCoy, R.A., *Baire spaces*, Dissert. Math. Warszawa 1977

[**5**] Künzi, H.P., *Topological spaces with a unique compatible quasi-proximity*, Arch. Math. **43** (1984), 559–561

[**6**] Künzi, H.P., *Topological spaces with a unique compatible quasi-uniformity*, Canad. Math. Bul. **29** (1986), 40–43

[**7**] Künzi, H.P., *Some remarks on quasi-uniform spaces*, Glasgow Math. J. **31** (1989), 309–320

[**8**] Lindgren, W.F., *Topological spaces with a unique compatible quasi-uniformity*, Canad. Math. Bul. **14** (1971), 369–372

[**9**] Pervin, W.J., *Quasi-uniformization of topological spaces*, Math. Ann. **147** (1962), 316–317

[**10**] Stone, A.H., *Hereditarily compact spaces*, Amer. J. Math. **82** (1960), 900–916

(received 10.04.1995.)

Jesús Ferrer
Dpt. Análisis Matemático
Facultad De Matematicas
46100 BURJASOT (Valencia), SPAIN

Valentin Gregori
E.U. Informatica
Universidad Politecnica De Valencia
Aptdo. 22012
46071 VALENCIA, SPAIN

Ivan L. Reilly
Department of Mathematics and Statistics
University of Auckland
Private Bag
AUCKLAND, NEW ZEALAND