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A CLASS OF GROWTH AND BOUNDS OF SOLUTIONS OF A DIFFERENTIAL EQUATION

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Abstract. We consider a class of nonlinear equations which admit characteristic equations. If the roots of this algebraic equation are real and distinct, the growth and bounds of solutions of the differential equation exist. An example is given which illustrates mentioned problems. An example of equation that is similar illustrates the problem of the existence of unique solutions.

THEOREM. *Consider the n-th order differential equation*

$$y^{(n)} + a_{n-1}(y(x))y^{(n-1)} + \cdots + a_1(y(x))y' + a_0(y(x))y = 0, \tag{1}$$

where $a_i(y(x))$, $0 \leqslant i \leqslant n-1$, are continuous functions. Let $\lambda_i(y(x))$, $1 \leqslant i \leqslant n$, be the roots of the equation $\lambda^n + a_{n-1}(y(x))\lambda^{n-1} + \cdots + a_0(y(x)) = 0$ and suppose that the functions $\lambda_i(y(x))$ are real valued and that there exist $2n$ constants $\alpha_1 \leqslant \beta_1 < \alpha_2 \leqslant \beta_2 < \cdots < \alpha_n \leqslant \beta_n$ such that

$$\alpha_i \leqslant \lambda_i(y(x)) \leqslant \beta_i \tag{2}$$

for $(x, y) \in [0, \omega) \times [0, \infty)$. Then (1) has n linearly independent solutions $y_1(x)$, $\dots$, $y_n(x)$ such that

$$y_i(x) > 0, \qquad \alpha_i \leqslant \frac{y_i'(x)}{y_i(x)} \leqslant \beta_i, \quad 1 \leqslant i \leqslant n,\ 0 \leqslant x \leqslant \omega. \tag{3}$$

If, additionally, (2) holds, then for each solution $y(x)$ of (1) there exist n solutions $y_1(x)$, $\dots$, $y_n(x)$ which satisfy (3) and n constants C_1, $\dots$, C_n, such that

$$y(x) = C_1 y_1(x) + \cdots + C_n y_n(x), \qquad 0 \leqslant x \leqslant \omega. \tag{4}$$

Proof. We recall a result of Hartman [1] which was for the first time proved in case $n = 2$ by Olech [3]. This result, together with (2), permits us to say that for each continuous function v from $[0, \omega)$ to $[0, \infty)$ the equation

$$y^{(n)} + a_{n-1}(v(x))y^{(n-1)} + \cdots + a_0(v(x))y = 0 \tag{5^v}$$

has n linearly independent solutions $y_i^v(x)$, $1 \leqslant i \leqslant n$, $x \in [0, \omega)$, which satisfy (3).

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Let Y be the space of continuous functions from $[0,\omega)$ into $\mathbf{R}$ with the compact-open topology (i.e. $y_m \to y$ means $\sup_{x\in J}|y_m(x)-y(x)| \to 0$, for each compact $J \subset [0,\omega)$). For $1 \leqslant i \leqslant n$ let

$$S_i = \{\, y \in Y : y(0)=1, e^{\alpha_i x} \leqslant y(x) \leqslant e^{\beta_i x}, 0 \leqslant x < \omega \,\}.$$

Each S_i is a closed, convex subset of Y. Fix i and let $S_i = S$. For $v \in S$ let

$$Tv = \{\, y \in S : y \text{ is a solution of } (5^v) \text{ which satisfies (3) for } 0 \leqslant x \leqslant \omega \,\}.$$

Then Tv is a non-empty, convex, compact subset of S. Hartman's theorem [1] shows that Tv is non-empty.

We shall apply the following fixed-point theorem [4]: Let S be a closed, convex, non-empty subset of a Banach space and let T satisfies: 1) for each $v \in S$, Tv is a compact, convex, non-empty subset of S; 2) if $v_m \in S$, $v_m \to v_0 \in S$ and $y_m \in T(v_m)$, $y_m \to y_0$, then $y_0 \in T(v_0)$; 3) TS is contained in a compact subset of S. Then there is a $v \in S$ such that $v \in Tv$. We need to verify 2) and 3).

2) Let J be a non-empty, compact subset of $[0,\omega)$. Since $J \subset [0,\gamma]$ for some $\gamma > 0$, we may assume that $J = [0,\gamma]$. Since $v_m \in S$, $y_m \in S$, $\{\, a_i(y_m(x)) : 0 \leqslant i \leqslant n-1 \,\}$, $\{v_m(x)\}$ and $\{y_m(x)\}$ are uniformly bounded on J. It follows that $\{y_m(x),\dots,y_m^{(n-1)}(x)\}$ and $\{y_m^{(n)}(x) = -\sum_{k=0}^{n-1} a_k(v_m(x))y_m^{(k)}(x)\}$ are uniformly bounded on J. By the Ascoli's theorem there is a subsequence $\{k\}$ of $\{m\}$ such that $(y_k(x),y_k'(x),\dots,y_k^{(n-1)}(x)) \to (y_0(x),z_1(x),\dots,z_{n-1}(x))$ uniformly on J as $k\to\infty$. Putting $\{y_k(x),\dots,y_k^{(n-1)}(x)\}$ and $\{v_k(x)\}$ into (5^v) we see that $\{y_k^{(n)}(x)\}$ also converges to $\{z_n(x)\}$, and that $y_0(x)$ is a solution of (5^{v_0}). Also, $y_0(x)$ satisfies (3).

3) It can be easily seen that $\overline{TS} \subset S$ is compact in X. If $\{y_n\}$ is a sequence in TS and $J \subset [0,\omega)$ is compact, then $\{y_m\}$ is uniformly bounded on J and, using (3), $\{y_m'\}$ is uniformly bounded on J. An application of Ascoli's theorem then shows that $\overline{TS}$ is compact.

Thus, the first part of the theorem is proved.

Suppose that $\bar{y}(x)$ is a solution of (1) and let $(\gamma,\delta) \subset [0,\omega)$ be its maximal interval of existence. Consider the linear differential equation

$$y^{(n)} + a_{n-1}(\bar{y}(x))y^{(n-1)} + \cdots + a_0(\bar{y}(x))y = 0. \tag{6}$$

From (2) and Hartman's theorem we have n functions $y_1, \dots, y_n(x)$ which satisfy (3) on (γ,δ) and which form a basis for the solution space of (6). Thus, for $\gamma < x < \delta$, $\bar{y}(x) = C_1y_1(x) + \cdots + C_ny_n(x)$, where $C_1, \dots, C_n$ are constants.

By standard arguments for prolonging solutions it follows that $(\gamma,\delta) = [0,\omega)$ and hence (4) follows. ■

EXAMPLE. Given the equation

$$y'' = 2(6 + a(x,y))y' + (20 + k \operatorname{arctg} y)y = 0, \tag{7}$$

where $a(x,y)$ is continuous and satisfies $|a(x,y)| \leqslant 1$, $(x,y) \in [0,\omega) \times \mathbf{R}$, for these values of the constant k, are we assured that solutions decay exponentially to zero? Physically, this equation may represent the motion of a nonlinear spring immersed in a liquid. The damping cannot be precisely measured, yet we want to be sure that the motion dies out.

One can verify that if $0 \leqslant k < 6/\pi$, then the theorem may be applied with

$$\begin{aligned} \alpha_1 &= -7 - \sqrt{29 + k\pi/2}, & \beta_1 &= -5 - \sqrt{5 - k\pi/2}, \\ \alpha_2 &= -7 + \sqrt{5 - k\pi/2}, & \beta_2 &= -5 + \sqrt{29 + k\pi/2} < 0. \end{aligned}$$

EXAMPLE. A standard method of solving the second order differential equation of the type

$$\frac{d^2y}{dx^2} + f(y) = 0, \qquad y(0) = C_1, \quad \frac{dy(0)}{dx} = C_2 \tag{8}$$

is to multiply the equation by dy/dx. Then it can be written in the form

$$\frac{1}{2}\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 + f(y)\frac{dy}{dx} = 0.$$

This last equation can be integrated to obtain

$$\left(\frac{dy}{dx}\right)^2 + F(y) = C_2^2 + F(C_1), \qquad y(0) = C_1, \tag{9}$$

where $dF(y)/dx = 2f(y)$. In this fashion, the second order equation (8) has been reduced to a first order equation (9). Presumably, once (9) has been solved, (8) has also been solved.

But it can happen that (8) is of a type which has the unique solution, whereas (9) does not. For example, the problem

$$\frac{d^2y}{dx^2} + y = 0, \qquad y(0) = 0, \quad \frac{dy}{dx}(0) = 1 \tag{10}$$

can be reduced to $(dy/dx)^2 + y^2 = 1$, $y(0) = 0$. One can easily check that each of the following is an acceptable solution of the last equation:

$$y_1 = \sin x,\ 0 \leqslant x < \infty, \qquad y_3 = \begin{cases} \sin x, & 0 \leqslant x < \pi/2, \\ 1, & \pi/2 \leqslant x < T, \\ \cos(x - T), & T \leqslant x < \infty. \end{cases}$$

$$y_2 = \begin{cases} \sin x, & 0 \leqslant x < \pi/2, \\ 1, & \pi/2 \leqslant x < \infty, \end{cases}$$

Only y_1, however, also satisfies (10). The reason for the multiplicity of solutions is related to the fact that whenever $y^2 = 1$, neither value of dy/dx from the reduced equation satisfies the Lipshitz condition. The question therefore rises, which additional conditions must be imposed to the reduced equation in order to be sure that selected solution also satisfies (10). We must select a solution of the reduced equation which is at least twice differentiable everywhere. The solution y_1 of the equation (10) and its reduced equation is evidently distinguished from the other solutions by this requirement. Actually, all solutions of (10) must be analytic.

This problem has much physical interest since in (9), $(dy/dx)^2$ can be considered as a kinetic energy term and $F(y)$ as a potential energy, so that (9) is a mathematical statement of the law of conservation of energy. (8) is the corresponding force equation.

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