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ON UNIFORM CONVERGENCE OF SPECTRAL EXPANSIONS AND THEIR DERIVATIVES ARISING BY SELF-ADJOINT EXTENSIONS OF AN ONE-DIMENSIONAL SCHRÖDINGER OPERATOR

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Abstract. In this paper we consider the problem of global uniform convergence of spectral expansions and their derivatives, $\sum_{n=1}^{\infty} f_n u_n^{(j)}(x)$ $(j = 0, 1, \dots)$, generated by non-negative self-adjoint extensions of the operator $\mathcal{L}(u)(x) = -u''(x) + q(x)u(x)$ with discrete spectrum, for functions from the class $W_2^{(1)}(G)$, where G is a finite interval of the real axis. Two theorems giving conditions on functions $q(x)$, $f(x)$ which are sufficient for the absolute and uniform convergence on $\overline{G}$ of the mentioned series, are proved. Also, some convergence rate estimates are obtained.

1. Introduction

1.1. On the problem. Let $G = (a, b)$ be a finite interval of the real axis $\mathbb{R}$. Consider an arbitrary non-negative self-adjoint extension L of the formal Schrödinger operator

$$\mathcal{L}(u)(x) = -u''(x) + q(x)u(x) \tag{1}$$

with a real-valued *non-negative* potential $q(x) \in L_1(G)$, defined by the self-adjoint boundary conditions

$$\begin{aligned} \alpha_{10}u(a) + \alpha_{11}u'(a) + \beta_{10}u(b) + \beta_{11}u'(b) = 0, \\ \alpha_{20}u(a) + \alpha_{21}u'(a) + \beta_{20}u(b) + \beta_{21}u'(b) = 0. \end{aligned} \tag{2}$$

(By this we mean a self-adjoint extension L of the corresponding symmetric operator L_0 in the sense of [2, §18]; the spectrum of such extension is discrete. Recall that the operator L is defined in the following way. Let $\mathcal{D}(L)$ be the set of functions $g(x) \in L_2(G)$ such that functions $g(x), g'(x)$ are absolutely continuous on $\overline{G}$, $\mathcal{L}(g)(x) \in L_2(G)$, and $g(x)$ satisfies the boundary conditions (2). If $g(x) \in \mathcal{D}(L)$, then $L(g)(x) \overset{\text{def}}{=} \mathcal{L}(g)(x)$.) Denote by $\{u_n(x)\}_1^\infty$ the orthonormal (and complete

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in $L_2(G)$) system of eigenfunctions of the extension L, and by $\{\lambda_n\}_1^\infty$ the corresponding system of non-negative eigenvalues enumerated in nondecreasing order. (By definition, $u_n(x) \in \mathcal{D}(L)$ and satisfies the differential equation

$$-u_n''(x) + q(x)u_n(x) = \lambda_n u_n(x) \tag{3}$$

almost everywhere on (a,b).)

Let $f(x) \in L_1(G)$ and let μ be an arbitrary positive number. We can form the partial sum of order μ of the expansion of $f(x)$ in terms of the system $\{u_n(x)\}_1^\infty$:

$$\sigma_\mu(x,f) \stackrel{\text{def}}{=} \sum_{\sqrt{\lambda_n}<\mu} f_n u_n(x),$$

where $f_n \stackrel{\text{def}}{=} \int_a^b f(x)u_n(x)\,dx$ are the Fourier coefficients of $f(x)$ relative to the system.

In this paper the classical problem of the absolute and uniform convergence on $\overline{G}$ of the functions $\sigma_\mu^{(j)}(x,f)$ $(j = 0,1,\dots)$, as $\mu \to +\infty$, is studied. We prove two theorems giving conditions on functions $q(x)$, $f(x)$ which are sufficient for the absolute and uniform convergence on $\overline{G}$ of the corresponding series. Also, we give some uniform, with respect to $x \in \overline{G}$, asymptotic estimates of the differences $f^{(j)}(x) - \sigma_\mu^{(j)}(x,f)$, as $\mu \to +\infty$.

1.2. Main results. We say that $f(x) \in W_p^{(k)}(G)$ $(1 \leqslant p < +\infty,\ k \in \mathbb{N})$ if $f(x) \in C^{(k-2)}(\overline{G})$, $f^{(k-1)}(x)$ is an absolutely continuous function on $[a,b]$ and $f^{(k)}(x) \in L_p(G)$. Also, $f(x) \in L_p(G)$ belongs to $H_p^\alpha(G)$ $(0 < \alpha \leqslant 1)$ if there is a constant $D(f) > 0$ such that

$$\|f(x+t) - f(x)\|_{L_p(G_{|t|})} \leqslant D(f)\cdot|t|^\alpha$$

for every $t \in (a-b, b-a)$, where $G_{|t|} = (a+|t|, b-|t|)$.

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function such that $f(a) = 0 = f(b)$. If $\sigma_\mu(x,f)$ converges *uniformly on* $[a,b]$, as $\mu \to +\infty$, then $f(x) = \lim_{\mu\to+\infty}\sigma_\mu(x,f)$ on $\overline{G}$ and, therefore,

$$\sigma_\mu(a,f) = o(1), \quad \sigma_\mu(b,f) = o(1), \qquad \mu \to +\infty. \tag{4}$$

So equalities (4) give necessary conditions for the uniform convergence of $\sigma_\mu(x,f)$ on $\overline{G}$. The following propositions describe some sufficient conditions for that convergence.

THEOREM 1. (*a*) *Let* $q(x) \in L_p(G)$ $(1 < p \leqslant 2)$, $q(x) \geqslant 0$, $f(x) \in W_2^{(1)}(G)$, *and* $f(a) = 0 = f(b)$. *Suppose that there exists a constant* $\mu_0 > 0$ *such that*

$$(\forall \mu \geqslant \mu_0) \qquad \mu^{1/2}\cdot\max\{|\sigma_\mu(a,f)|, |\sigma_\mu(b,f)|\} \leqslant D(f,q), \tag{5}$$

where $D(f,q) \stackrel{\text{def}}{=} \max\{\lambda_1 f_1^2\big(8AC_0C_1\int_a^b |f(x)|\,dx\big)^{-1}, \lambda_1^{1/2}|f_1|(8C_1)^{-1}\}$; A, C_0, C_1 *are constants from Propositions* 1 *and* 2. *Then for every* $x \in \overline{G}$ *the equality*

$$f(x) = \sum_{n=1}^\infty f_n u_n(x) \tag{6}$$

holds, and the series converges absolutely and uniformly on $\overline{G}$.

Also, the following estimate is valid:

$$\max_{x\in\overline{G}}\big|f(x)-\sigma_\mu(x,f)\big| = O\Big(\frac{1}{\mu^{1/2}}\Big). \tag{7}$$

(*b*) *Suppose* $q(x)\in W_2^{(1)}(G)$, $q(x)\geqslant 0$, $f(x)\in W_2^{(3)}(G)$, $f(x)$ *satisfies the boundary conditions* (2), *and* $\mathcal{L}(f)(a)=0=\mathcal{L}(f)(b)$. *If there is a constant* $\mu_1>0$ *such that*

$$(\forall\mu\geqslant\mu_1)\mu^{1/2}\cdot\max\{|\sigma_\mu(a,\mathcal{L}(f))|,|\sigma_\mu(b,\mathcal{L}(f))|\}\leqslant D(\mathcal{L}(f),q), \tag{8}$$

where $D(\mathcal{L}(f),q)$ *is defined analogously to* $D(f,q)$, *then the equalities*

$$f^{(j)}(x)=\sum_{n=1}^{\infty} f_n u_n^{(j)}(x), \qquad 0\leqslant j\leqslant 2, \tag{9}$$

are valid on $\overline{G}$, *and the series converge absolutely and uniformly on* $\overline{G}$.

Moreover, the following estimates hold:

$$\max_{x\in\overline{G}}\big|f^{(j)}(x)-\sigma_\mu^{(j)}(x,f)\big| = O\Big(\frac{1}{\mu^{5/2-j}}\Big), \quad 0\leqslant j\leqslant 2. \tag{10}$$

In the case of a special but important class of boundary conditions (2), it turned out that conditions (5) and (8) can be weakened essentially. Namely, the following is valid.

THEOREM 2. *Let us suppose that*

$$\alpha_{11}\beta_{21}-\alpha_{21}\beta_{11}\neq 0. \tag{11}$$

Then: (*a*) *Proposition* (*a*) *of Theorem* 1 *is still valid, if condition* (5) *is relaxed by*

$$\sigma_\mu(a,f)=O(1), \qquad \sigma_\mu(b,f)=O(1). \tag{12}$$

(*b*) *Proposition* (*b*) *of Theorem* 1 *holds, if condition* (8) *is replaced by*

$$\sigma_\mu(a,\mathcal{L}(f))=O(1), \qquad \sigma_\mu(b,\mathcal{L}(f))=O(1). \tag{12*}$$

REMARK 1. In the case of an *arbitrary* self-adjoint extension L of the operator (1), the following results were established in papers [6] and [8]:

Let $q(x)\in L_2(G)$, $f(x)\in W_2^{(2)}(G)$. If $f(x)$ satisfies the boundary conditions (2), then the equalities

$$f^{(j)}(x)=\sum_{n=1}^{\infty} f_n u_n^{(j)}(x), \qquad 0\leqslant j\leqslant 1,$$

are valid on $\overline{G}$, and the series converge absolutely and uniformly on $\overline{G}$. Moreover, the following estimates hold:

$$\max_{x\in\overline{G}}\big|f^{(j)}(x)-\sigma_\mu^{(j)}(x,f)\big| = o\Big(\frac{1}{\mu^{3/2-j}}\Big), \quad 0\leqslant j\leqslant 1.$$

REMARK 2. In the case of an *arbitrary* self-adjoint extension L of the operator (1), the problem we stady was considered in papers [6] and [8] for the most important subclasses of $W_2^{(1)}(G)$. By a method differing from the one used in this paper, we proved in paper [6] that the series (6) converge absolutely and uniformly on $\overline{G}$ (and equality (6) holds) under the assumptions

(a) $q(x) \in L_p(G)$ $(1 < p \leqslant 2)$, $f(x) \in W_1^{(1)}(G)$, $f(a) = 0 = f(b)$ and $f'(x)$ is a bounded, piecewise monotone function on its domain $\mathcal{D}(f') \subseteq \overline{G}$ or $f'(x) \in BV(\overline{G})$,

and that the same is true for the series (9) if

(b) $q(x) \in AC(\overline{G})$, $f(x) \in W_1^{(3)}(G)$ and satisfies the boundary conditions (2), $\mathcal{L}(f)(a) = 0 = \mathcal{L}(f)(b)$, and $\mathcal{L}(f)'(x)$ is a bounded, piecewise monotone function on its domain $\mathcal{D}(\mathcal{L}(f)') \subseteq \overline{G}$ or $\mathcal{L}(f)'(x) \in BV(\overline{G})$.

Here $AC(\overline{G})$ denotes the class of absolutely continuous functions on the closed interval $\overline{G} = [a, b]$, and $BV(\overline{G})$ is the class of functions having the bounded variation on this interval. Let us note that a real valued function g, defined on a set $\mathcal{D}(g) \subseteq \overline{G}$, is called *piecewise monotone* on $\mathcal{D}(g)$ if there is a set $\{x_0, x_1, \dots, x_{n(g)}\} \subset \overline{G}$ such that $a = x_0 < x_1 < \cdots < x_{n(g)} = b$ and functions $g\,|_{\mathcal{D}(g)\cap[x_{i-1},x_i]}$ are monotone for every $i = 1, \dots, n(g)$.

In paper [8] the above results were completed by the estimate

$$\max_{x\in\overline{G}}\bigl|f(x) - \sigma_\mu(x, f)\bigr| = O\left(\frac{1}{\mu}\right), \tag{13}$$

which holds in the case of assumptions (a), and by the estimates

$$\max_{x\in\overline{G}}\bigl|f^{(j)}(x) - \sigma_\mu^{(j)}(x, f)\bigr| = O\left(\frac{1}{\mu^{3-j}}\right), \quad 0 \leqslant j \leqslant 2,$$

which are valid in the case of assumptions (b).

In paper [8] we also proved the following propositions.

(c) Let $q(x) \in L_p(G)$ $(1 < p \leqslant 2)$, $f(x) \in W_1^{(1)}(G)$ and $f'(x) \in L_\infty(G) \cap H_1^\alpha(G)$, $0 < \alpha \leqslant 1$. If $f(a) = 0 = f(b)$, then series (6) converges absolutely and uniformly on $\overline{G}$ and the following estimate holds:

$$\max_{x\in\overline{G}}\bigl|f(x) - \sigma_\mu(x, f)\bigr| = O\left(\frac{1}{\mu^\alpha}\right) + o\left(\frac{1}{\mu^{1/2}}\right). \tag{14}$$

(d) Suppose $q(x) \in W_2^{(1)}(G)$, $f(x) \in W_2^{(3)}(G)$ and $\mathcal{L}(f)'(x)$ belongs to the class $L_\infty(G) \cap H_1^\alpha(G)$, $0 < \alpha \leqslant 1$. If $f(x)$ satisfies the boundary conditions (2) and $\mathcal{L}(f)(a) = 0 = \mathcal{L}(f)(b)$, then the equalities (9) hold on $\overline{G}$, the series being absolutely and uniformly convergent on $\overline{G}$, and the estimates

$$\max_{x\in\overline{G}}\bigl|f^{(j)}(x) - \sigma_\mu^{(j)}(x, f)\bigr| = O\left(\frac{1}{\mu^{2-j+\alpha}}\right) + o\left(\frac{1}{\mu^{2-j+1/2}}\right)$$

are valid, where $0 \leqslant j \leqslant 2$.

REMARK 3. Considering the classes of functions defined by conditions (5) or (8), we can say the following. Condition (5) is satisfied if, especially,

$$\sigma_\mu(a,f) = o\left(\frac{1}{\mu^{1/2}}\right), \qquad \sigma_\mu(b,f) = o\left(\frac{1}{\mu^{1/2}}\right).$$

Therefore, it results from the estimate (13) that functions described by (a) in the preceding remark belong to the first class. Also, if $f(x)$ belongs to the class defined in (c), where $\alpha > 1/2$, then by virtue of estimate (14) this function satisfies condition (5).

As far as condition (8) concerned, the functions $f(x)$ described in (b) or (d) (with $\alpha > 1/2$ in the latter case) satisfy this condition. Namely, in these cases it is possible to apply estimate (13) or (14) respectively to the function $\mathcal{L}(f)(x)$.

REMARK 4. It is possible to obtain results concerning the absolute and uniform convergence on $\overline{G}$ of higher derivatives $\sigma_\mu^{(j)}(x,f)$, where $j \geqslant 3$. We omit here the corresponding theorem, but note that this program was realised in [8, Theorem 2] for classes of functions mentioned in Remark 2.

REMARK 5. Let the boundary conditions (2) have one of the following three forms:

$$1)\ u(a) = 0 = u(b); \qquad 2)\ u'(a) = 0 = u'(b); \qquad 3)\ u(a) = u(b), u'(a) = u'(b).$$

Then conditions (5) and (8) can be omitted (see section 4.3).

Note that, in the case 1), proposition (a) of Theorem 1 was first established in [1, Lemma 7].

Our approach to the problem considered is based only on uniform and exact (with respect to order) estimates for the moduli of eigenfunctions and their derivatives (see Propositions 1–2 bellow). The results obtained may have a theoretical interest on their own. From the point of view of "applications", Theorem 1 and results stated in Remark 1 may allow us to prove the existence and uniqueness of classical solutions to a large class of "self-adjoint" mixed boundary problems for one-dimensional hyperbolic or parabolic equations of second order. In the hyperbolic case, for example, these problems have the form:

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) + q(x)u(x,t) = f(x,t), \quad (x,t) \in G \times (0,T),$$
$$u(x,0) = \varphi(x), \quad u'_t(x,0) = \psi(x), \qquad x \in \overline{G},$$

$$\alpha_{10}u(a,t) + \alpha_{11}u'_x(a,t) + \beta_{10}u(b,t) + \beta_{11}u'_x(b,t) = 0,$$
$$\alpha_{20}u(a,t) + \alpha_{21}u'_x(a,t) + \beta_{20}u(b,t) + \beta_{21}u'_x(b,t) = 0, \quad t \in [0,T],$$

where $T > 0$ is an arbitrary number. Note that, following this line, in papers [5]–[7] we have established the existence and uniqueness of classical solutions to the mentioned boundary problems for classes of functions cited in propositions (a) and (b) from Remark 2.

1.3. Auxiliary propositions. In proofs of the theorems we essentially use known estimates concerning the eigenvalues and eigenfunctions (and their derivatives) of the operator (1), which have been obtained by several authors.

Let $\{u_n(x)\}_1^\infty$ be the orthonormal system of eigenfunctions arising by an *arbitrary* non-negative self-adjoint extension L of the operator (1) with a potential $q(x) \in L_1(G)$, and let $\{\lambda_n\}_1^\infty$ be the corresponding system of non-negative eigenvalues enumerated in nondecreasing order. Then the following propositions hold.

PROPOSITION 1 ([3]). (*a*) If $q(x) \in L_1(G)$, then there exists a constant $C_0 > 0$, independent of $n \in \mathbb{N}$, such that

$$\max_{x \in \overline{G}} |u_n(x)| \leqslant C_0, \qquad n \in \mathbb{N}. \tag{15}$$

(*b*) If $q(x) \in L_p(G)(p > 1)$, then there exists a constant $A > 0$ such that

$$\sum_{t \leqslant \sqrt{\lambda_n} \leqslant t+1} 1 \leqslant A \tag{16}$$

for every $t \geqslant 0$, where A does not depend on t.

PROPOSITION 2 ([4]). (*a*) If $q(x) \in L_1(G)$, then there exist constants $\mu_0 = \mu_0(G) > 0$ and $C_1 > 0$, not depending on $n \in \mathbb{N}$, such that

$$\max_{x \in \overline{G}} |u_n'(x)| \leqslant \begin{cases} C_1 \sqrt{\lambda_n} & \text{if} \quad \lambda_n > \mu_0, \\ C_1 & \text{if} \quad 0 \leqslant \lambda_n \leqslant \mu_0. \end{cases} \tag{17}$$

(*b*) Suppose $q(x) \in \mathcal{C}^{(k-2)}(\overline{G})$ $(k \geqslant 2)$. Then $u_n(x) \in \mathcal{C}^{(k)}(\overline{G})$, and there exist constants $C_j > 0 (2 \leqslant j \leqslant k)$, independent of $n \in \mathbb{N}$, such that

$$\max_{x \in \overline{G}} |u_n^{(j)}(x)| \leqslant \begin{cases} C_j \lambda_n^{j/2} & \text{if} \quad \lambda_n > \mu_0, \\ C_j & \text{if} \quad 0 \leqslant \lambda_n \leqslant \mu_0. \end{cases} \tag{18}$$

Note that the constants A, C_i $(i = 0, 1, \dots)$ depend on G and $q(x)$. For the sake of simplicity the estimates (17)–(18) will be used supposing that $\mu_0 = 1$.

Estimates (15)–(18) allow us to prove the following assertions, which play the central role in proving Theorems 1 and 2.

PROPOSITION 3. (*a*) Suppose $q(x) \in L_p(G)$ $(1 < p \leqslant 2)$, $q(x) \geqslant 0$, $f(x) \in W_2^{(1)}(G)$, $f(a) = 0 = f(b)$, and condition (5) is satisfied. Then there exists a number $n_0 \in \mathbb{N}$ such that for every $n > n_0$ the following inequality holds:

$$\sum_{k=1}^{n} \lambda_k f_k^2 \leqslant 2 \cdot \int_a^b \left[(f'(x))^2 + q(x) f^2(x) \right] dx. \tag{19}$$

(*b*) Let $q(x) \in L_2(G)$, $f(x) \in W_2^{(2)}(G)$, and $f(x)$ satisfy the boundary conditions (2). Then for every $n \in \mathbb{N}$ the following inequality holds:

$$\sum_{k=1}^{n} \lambda_k^2 f_k^2 \leqslant \int_a^b \left(\mathcal{L}(f)(x) \right)^2 dx. \tag{20}$$

(*c*) Suppose $q(x) \in W_2^{(1)}(G)$, $q(x) \geqslant 0$, $f(x) \in W_2^{(3)}(G)$ and satisfies the boundary conditions (2), $\mathcal{L}(f)(a) = 0 = \mathcal{L}(f)(b)$, and condition (8) is satisfied. Then there is a number $n_1 \in \mathbb{N}$ such that for every $n > n_1$ the following inequality holds:

$$\sum_{k=1}^{n} \lambda_k^3 f_k^2 \leqslant 2 \cdot \int_a^b \big[\big(\mathcal{L}(f)'(x)\big)^2 + q(x)\big(\mathcal{L}(f)(x)\big)^2 \big]\, dx. \tag{21}$$

2. Proof of proposition 3

2.1. Proposition 3(a). Proposition 3 (and a part of its proof) is modeled according to Lemma 3 from paper [1].

Let the functions $q(x)$, $f(x)$ satisfy conditions imposed in Proposition 3(a). Suppose $\lambda_n \neq 0$. Using differential equation (3) and the integration by parts, we obtain the equalities

$$\begin{aligned}\lambda_n \cdot \int_a^b f(x) u_n(x)\, dx &= \int_a^b f(x)\big[-u_n''(x) + q(x)u_n(x)\big]\, dx \\ &= -u_n'(x)f(x)\big|_a^b + \int_a^b \big[u_n'(x)f'(x) + q(x)u_n(x)f(x)\big]\, dx, \end{aligned} \tag{22}$$

wherefrom it follows, by $f(a) = f(b) = 0$, that

$$\int_a^b \big[u_n'(x)f'(x) + q(x)u_n(x)f(x)\big]\, dx = \lambda_n f_n. \tag{23}$$

Replacing $f(x)$ by $u_m(x)$ in (22), one can obtain the equality

$$\begin{aligned}\lambda_n \cdot \int_a^b & u_m(x)u_n(x)\, dx \\ &= -u_n'(b)u_m(b) + u_n'(a)u_m(a) + \int_a^b \big[u_n'(x)u_m'(x) + q(x)u_n(x)u_n(x)\big]\, dx,\end{aligned}$$

and then, by the orthonormallity of the system $\{u_n(x)\}_1^\infty$, that the following equalities hold:

$$\begin{aligned}\int_a^b & \big[u_n'(x)u_m'(x) + q(x)u_n(x)u_m(x)\big]\, dx \\ &= \begin{cases} u_n'(b)u_n(b) - u_n'(a)u_n(a) + \lambda_n & \text{if} \quad m = n, \\ u_n'(b)u_m(b) - u_n'(a)u_m(a) & \text{if} \quad m \neq n. \end{cases}\end{aligned} \tag{24}$$

Let us now introduce the integral

$$\begin{aligned} 0 \leqslant\ & I_n(q, f, G) \\ &\stackrel{\text{def}}{=} \int_a^b \Bigg[\Big(f'(x) - \sum_{k=1}^{n} f_k u_k'(x) \Big)^2 + q(x) \Big(f(x) - \sum_{k=1}^{n} f_k u_k(x) \Big)^2 \Bigg]\, dx,\end{aligned}$$

where $n \in \mathbb{N}$ is an arbitrary number. After the obvious transformations the integral can obtain the following form:

$$\begin{aligned} I_n(q,f,G) = \int_a^b \left[(f'(x))^2 + q(x)f^2(x)\right] dx - \\ - 2 \cdot \sum_{k=1}^{n} \Big(\int_a^b \left[u_k'(x)f'(x) + q(x)u_k(x)f(x)\right] dx \Big) \cdot f_k + \\ + \int_a^b \Big[\Big(\sum_{k=1}^{n} f_k u_k'(x) \Big)^2 + q(x) \Big(\sum_{k=1}^{n} f_k u_k(x) \Big)^2 \Big] dx. \end{aligned}$$

Using equality (23), we have

$$\begin{aligned} I_n(q,f,G) &= \int_a^b \left[(f'(x))^2 + q(x)f^2(x)\right] dx - 2 \cdot \sum_{k=1}^{n} \lambda_k f_k^2 + \\ &\quad + \int_a^b \Big[\Big(\sum_{k=1}^{n} f_k u_k'(x) \Big)^2 + q(x) \Big(\sum_{k=1}^{n} f_k u_k(x) \Big)^2 \Big] dx \\ &= \int_a^b \left[(f'(x))^2 + q(x)f^2(x)\right] dx - 2 \cdot \sum_{k=1}^{n} \lambda_k f_k^2 + \\ &\quad + \sum_{k=1}^{n} f_k^2 \cdot \int_a^b \left[u_k'(x)u_k'(x) + q(x)u_k(x)u_k(x)\right] dx + \\ &\quad + 2 \cdot \sum_{k=1}^{n-1} \sum_{k<j}^{n} f_k f_j \cdot \int_a^b \left[u_k'(x)u_j'(x) + q(x)u_k(x)u_j(x)\right] dx. \end{aligned}$$

It remains to apply equalities (24) to the last two integrals:

$$\begin{aligned} I_n(q,f,G) = \int_a^b \left[(f'(x))^2 + q(x)f^2(x)\right] dx - \sum_{k=1}^{n} \lambda_k f_k^2 + \\ + \sum_{k=1}^{n} f_k^2 \big(u_k'(b)u_k(b) - u_k'(a)u_k(a)\big) dx + \\ + 2 \cdot \sum_{k=1}^{n-1} \sum_{k<j}^{n} f_k f_j \big(u_k'(b)u_j(b) - u_k'(a)u_j(a)\big) dx. \end{aligned}$$

We know that $u_k(x), u_j(x) \in \mathcal{D}(L)$ and L is a self-adjoint operator. So we have that

$$-u_k'(x)u_j(x)\Big|_a^b + u_k(x)u_j'(x)\Big|_a^b = 0, \qquad \text{or}$$
$$u_k'(b)u_j(b) - u_k'(a)u_j(a) = u_k(b)u_j'(b) - u_k(a)u_j'(a).$$

By the force of this equality, $I_n(q,f,G)$ can be transformed further:

$$I_n(q,f,G)=\int_a^b\left[(f'(x))^2+q(x)f^2(x)\right]dx-\sum_{k=1}^n\lambda_k f_k^2+$$
$$+\left(\sum_{k=1}^n f_k u_k'(b)\right)\left(\sum_{k=1}^n f_k u_k(b)\right)-\left(\sum_{k=1}^n f_k u_k'(a)\right)\left(\sum_{k=1}^n f_k u_k(a)\right).$$

Therefore, having in mind the non-negativity of $I_n(q,f,G)$, we obtain the following inequality:

$$\sum_{k=1}^n\lambda_k f_k^2\leqslant\int_a^b\left[(f'(x))^2+q(x)f^2(x)\right]dx+$$
$$+\left(\sum_{k=1}^n f_k u_k'(b)\right)\left(\sum_{k=1}^n f_k u_k(b)\right)-\left(\sum_{k=1}^n f_k u_k'(a)\right)\left(\sum_{k=1}^n f_k u_k(a)\right). \quad (25)$$

Let us estimate the above products of two sums. If $j\in\mathbb{N}$ is such that $1\leqslant j<n$, $\lambda_k\leqslant 1$ for $1\leqslant k\leqslant j$, and $\lambda_k>1$ for $j<k\leqslant n$ (we suppose that n is big enough), then by estimates (15)–(17) and the Cauchy-Schwartz inequality we can obtain

$$\left|\sum_{k=1}^n f_k u_k'(b)\right|\leqslant\sum_{k=1}^j|f_k u_k'(b)|+\sum_{k=j+1}^n|f_k u_k'(b)|$$
$$\leqslant AC_0C_1\cdot\int_a^b|f(x)|\,dx+C_1\cdot\sum_{k=j+1}^n\sqrt{\lambda_k}|f_k|$$
$$\leqslant D_1+C_1\cdot\sum_{k=1}^n\sqrt{\lambda_k}|f_k|\leqslant D_1+C_1\left(\sum_{k=1}^n\lambda_k f_k^2\right)^{1/2}\cdot n^{1/2},$$

where D_1 has an obvious meaning. Further we have

$$\left(\sum_{k=1}^n\lambda_k f_k^2\right)^{1/2}=\sum_{k=1}^n\lambda_k f_k^2\cdot\left(\sum_{k=1}^n\lambda_k f_k^2\right)^{-1/2}\leqslant\frac{1}{\lambda_1^{1/2}|f_1|}\cdot\sum_{k=1}^n\lambda_k f_k^2.$$

(We can suppose, with no loss of generality, that $\lambda_1|f_1|\neq 0$.) Therefore, we can conclude that the following estimate holds:

$$\left|\left(\sum_{k=1}^n f_k u_k'(b)\right)\left(\sum_{k=1}^n f_k u_k(b)\right)\right|$$
$$\leqslant D_1|\sigma_n(b,f)|+\frac{C_1}{\lambda_1^{1/2}|f_1|}\left(\sum_{k=1}^n\lambda_k f_k^2\right)\cdot n^{1/2}|\sigma_n(b,f)|, \quad (26)$$

where $\sigma_n(x,f)\overset{\text{def}}{=}\sum_{k=1}^n f_k u_k(x)$. Analogously, we have the estimate

$$\left|\left(\sum_{k=1}^n f_k u_k'(a)\right)\left(\sum_{k=1}^n f_k u_k(a)\right)\right|$$
$$\leqslant D_1|\sigma_n(a,f)|+\frac{C_1}{\lambda_1^{1/2}|f_1|}\left(\sum_{k=1}^n\lambda_k f_k^2\right)\cdot n^{1/2}|\sigma_n(a,f)|. \quad (27)$$

Now, we have to use assumption (5). It can be restated in the equivalent form: there is a number $n_0 \in \mathbb{N}$ such that for each entire number $n > n_0$ we have

$$n^{1/2} \cdot \max\{|\sigma_n(a,f)|, |\sigma_n(b,f)|\} \leqslant D(f,q).$$

Therefore, for every $n > n_0$ the following estimates hold:

$$\begin{gathered} D_1 \cdot \max\{|\sigma_n(a,f)|, |\sigma_n(b,f)|\} < \frac{\lambda_1 f_1^2}{8} < \frac{1}{8}\sum_{k=1}^{n} \lambda_k f_k^2, \\ \frac{C_1}{\lambda_1^{1/2}|f_1|} \cdot n^{1/2} \cdot \max\{|\sigma_n(a,f)|, |\sigma_n(b,f)|\} < \frac{1}{8}. \end{gathered} \tag{28}$$

Finally, from (25)–(28) it results that for every $n > n_0$ the estimate (19) is valid:

$$\sum_{k=1}^{n} \lambda_k f_k^2 \leqslant 2 \cdot \int_a^b \left[(f'(x))^2 + q(x) f^2(x)\right] dx.$$

Proof of Proposition 3(a) is completed.

2.2. Proposition 3(b). This proposition is concerned with results formulated in Remark 1, and it is not needed in proofs of our theorems. But for the sake of completeness of the exposition we have stated the proposition and will give now its proof.

In this case, the conditions imposed on functions $q(x)$, $f(x)$ imply that $\mathcal{L}(f)(x) \in L_2(G)$, so $f(x)$ belongs to the domain of the operator L considered. Also, we can use the integration by parts and transforme the integral appearing on the right-hand side of the second equality (22):

$$\lambda_n f_n = -u_n'(x) f(x)\big|_a^b + u_n(x) f'(x)\big|_a^b + \int_a^b \left[-f''(x) + q(x) f(x)\right] u_n(x)\, dx.$$

But $-u_n'(x) f(x)\big|_a^b + u_n(x) f'(x)\big|_a^b = 0$ because of the self-adjointness of L. Therefore we have the equalities

$$\lambda_n f_n = \int_a^b \mathcal{L}(f)(x) u_n(x)\, dx = \mathcal{L}(f)_n. \tag{29}$$

Now, by the Bessel inequality, for every $n \in \mathbb{N}$ the following holds:

$$\sum_{k=1}^{n} \mathcal{L}(f)_n^2 \leqslant \int_a^b \left(\mathcal{L}(f)(x)\right)^2 dx.$$

But this means, by virtue of equalities (29), that inequality (20) is valid.

2.3. Proposition 3(c). Let the functions $q(x)$ and $f(x)$ satisfy conditions imposed in the proposition. Then for function $\mathcal{L}(f)(x)$ all the assumptions from

Proposition 3(a) are fulfilled. So we can apply the inequality (19): there is a number $n_1 \in \mathbb{N}$ such that for each entire number $n > n_1$ the inequality

$$\sum_{k=1}^{n} \lambda_k \mathcal{L}(f)_k^2 \leqslant 2 \cdot \int_a^b \left[\left(\mathcal{L}(f)'(x)\right)^2 + q(x)\left(\mathcal{L}(f)(x)\right)^2\right] dx \tag{30}$$

holds. Now, let us recall the equalities (29): replacing $\mathcal{L}(f)_k$ by $\lambda_k f_k$ in (30), we obtain the inequality (21).

Proof of Proposition 3 is completed. ■

3. Proof of Theorem 1

3.1. Proposition (a). Let us establish first that the series (6) converges absolutely and uniformly on $\overline{G}$. If $n_0 \in \mathbb{N}$ is the number from Proposition 3(a) and $j \in \mathbb{N}$ is such that $\lambda_k \leqslant 1$ for $1 \leqslant k \leqslant j$, and $\lambda_n > 1$ for each $k > j$, then we can suppose, with no loss of generality, that $j < n_0$. Now, let $n > n_0$ be an arbitrary entire number. By estimates (15)–(16) and inequality (19), for every $x \in \overline{G}$ we have

$$\begin{aligned}
\sum_{k=1}^{n} |f_k u_k(x)| &= \sum_{k=1}^{j} |f_k u_k(x)| + \sum_{k=j+1}^{n} |f_k u_k(x)| \\
&\leqslant AC_0^2 \cdot \int_a^b |f(x)|\, dx + C_0 \cdot \sum_{k=j+1}^{n} |f_k| \\
&\leqslant D_2 + C_0 \Big(\sum_{k=j+1}^{n} \lambda_k f_k^2 \Big)^{1/2} \Big(\sum_{k=j+1}^{n} \frac{1}{\lambda_k} \Big)^{1/2} \\
&\leqslant D_2 + C_0 \Big(2 \int_a^b \left[(f'(x))^2 + q(x) f^2(x) \right] dx \Big)^{1/2} \Big[\sum_{i=1}^{\infty} \Big(\sum_{i < \sqrt{\lambda_k} \leqslant i+1} \frac{1}{\lambda_k} \Big) \Big]^{1/2} \\
&\leqslant D_2 + D_3 A^{1/2} \Big(\sum_{i=1}^{\infty} \frac{1}{i^2} \Big)^{1/2},
\end{aligned}$$

whereform the absolute and uniform convergence of series (6) on $\overline{G}$ follows.

The equality (6) results from the continuity of $f(x)$ and the completeness of the system $\{u_n(x)\}_1^\infty$ in $L_2(G)$.

It remains to prove estimate (7). Having equality (6) established, for any $x \in \overline{G}$ and $\mu \geqslant 2$ we can write

$$f(x) - \sigma_\mu(x, f) = \sum_{\sqrt{\lambda_k} \geqslant \mu} f_k u_k(x). \tag{31}$$

Let $n(\mu) \in \mathbb{N}$ be the smallest number such that $\mu \leqslant \sqrt{\lambda_{n(\mu)}}$, and let $n > \max\{n_0, n(\mu)\}$ be an arbitrary entire number. Then, by estimates (15)–(16) and inequality (19), as above we have

$$\sum_{k=n(\mu)}^{n} |f_k u_k(x)| \leqslant C_0 \Big(\sum_{k=n(\mu)}^{n} \lambda_k f_k^2\Big)^{1/2} \Big(\sum_{k=n(\mu)}^{n} \frac{1}{\lambda_k}\Big)^{1/2}$$
$$\leqslant D_3 \Big(\sum_{k=n(\mu)}^{\infty} \frac{1}{\lambda_k}\Big)^{1/2} \leqslant D_3 \Big[\sum_{i=[\mu]}^{\infty} \Big(\sum_{i\leqslant\sqrt{\lambda_k}<i+1} \frac{1}{\lambda_k}\Big)\Big]^{1/2}$$
$$\leqslant D_3 A^{1/2} \Big(\sum_{i=[\mu]}^{\infty} \frac{1}{i^2}\Big)^{1/2} \leqslant D_3 A^{1/2} \Big(\int_{[\mu]-1}^{+\infty} \frac{dt}{t^2}\Big)^{1/2} \leqslant D_3 (2A)^{1/2} \cdot \frac{1}{\mu^{1/2}},$$

wherefrom it follows that the estimate

$$\Big|\sum_{\sqrt{\lambda_k}\geqslant\mu} f_k u_k(x)\Big| = O\Big(\frac{1}{\mu^{1/2}}\Big) \tag{32}$$

holds uniformly with respect to $x \in \overline{G}$. Therefore, by virtue of (31)–(32), we conclude that estimate (7) is valid.

3.2. Proposition (b). We already have a model for the proof of this proposition. So, let $n_1 \in \mathbb{N}$ be the number from Proposition 3(c), and let $1 \leqslant j < n_0$ be defined as in the preceding section. Suppose $n > n_1$ and $x \in \overline{G}$ are arbitrary. Then for each $r \in \{0,1,2\}$ we have

$$\sum_{k=1}^{n} |f_k u_k^{(r)}(x)| = \sum_{k=1}^{j} |f_k u_k^{(r)}(x)| + \sum_{k=j+1}^{n} |f_k u_k^{(r)}(x)|$$
$$\leqslant AC_0C_r \cdot \int_a^b |f(x)|\,dx + C_r \cdot \sum_{k=j+1}^{n} \lambda_k^{r/2} |f_k|$$
$$\leqslant D_4 + C_r \Big(\sum_{k=j+1}^{n} \lambda_k^3 f_k^2\Big)^{1/2} \Big(\sum_{k=j+1}^{n} \frac{1}{\lambda_k^{3-r}}\Big)^{1/2} \leqslant D_4 +$$
$$+ C_r \Big(2\int_a^b \big[\big(\mathcal{L}(f)'(x)\big)^2 + q(x)\big(\mathcal{L}(f)(x)\big)^2\big]\,dx\Big)^{1/2} \Big[\sum_{i=1}^{\infty} \Big(\sum_{i<\sqrt{\lambda_k}\leqslant i+1} \frac{1}{\lambda_k^{3-r}}\Big)\Big]^{1/2}$$
$$\leqslant D_4 + D_5 A^{1/2} \Big(\sum_{i=1}^{\infty} \frac{1}{i^{2(3-r)}}\Big)^{1/2},$$

where the constants D_4, D_5 have an obvious meaning. Hence, we see that the series (9) converge absolutely and uniformly on $\overline{G}$. Now, the equalities (9) can be established by the classical theorem on differentiability of uniformly convergent functional series.

Let us prove estimates (10). By equalities (9), we can write

$$f^{(r)}(x) - \sigma_\mu^{(r)}(x,f) = \sum_{\sqrt{\lambda_k}\geqslant\mu} f_k u_k^{(r)}(x), \tag{33}$$

for all $x \in \overline{G}$ and $0 \leqslant r \leqslant 2$. Let $\mu \geqslant 2$ be fixed, and let $n(\mu) \in \mathbb{N}$ be defined as in the preceding section. So if $x \in \overline{G}$ and $n > \max\{n_1, n(\mu)\}$ are arbitrary, then we have

$$\sum_{k=n(\mu)}^{n} |f_k u_k^{(r)}(x)| \leqslant C_r \sum_{k=n(\mu)}^{n} \lambda_k^{r/2}|f_k| \leqslant C_r \Big(\sum_{k=n(\mu)}^{n} \lambda_k^3 f_k^2\Big)^{1/2} \Big(\sum_{k=n(\mu)}^{n} \frac{1}{\lambda_k^{3-r}}\Big)^{1/2}$$
$$\leqslant D_5 \Big[\sum_{i=[\mu]}^{\infty} \Big(\sum_{i \leqslant \sqrt{\lambda_k} < i+1} \frac{1}{\lambda_k^{3-r}}\Big)\Big]^{1/2} \leqslant D_5 A^{1/2} \Big(\sum_{i=[\mu]}^{\infty} \frac{1}{i^{6-2r}}\Big)^{1/2}$$
$$\leqslant D_5 A^{1/2} \Big(\int_{[\mu]-1}^{+\infty} \frac{dt}{t^{6-2r}}\Big)^{1/2} \leqslant D_6 \cdot \frac{1}{\mu^{(5-2r)/2}}.$$

(Here we used estimates (15)–(18) and inequality (21).) From the above inequalities and (33) it results that estimates (10) hold.

Proof of Theorem 1 is completed. ■

4. Proof of Theorem 2

4.1. On Proposition 3. The central point in the proof of Theorem 2 is based on the following observation: Under assumption (11), Propositions 3(a) and 3(c) can be appropriately modified, if conditions (5) and (8) are replaced by conditions (12) and (12*) respectively.

Proposition 3 (a) will be considered first. Let the coefficients α_{ij}, β_{ij} from boundary conditions (2) satisfy (11):

$$\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11} \neq 0.$$

Then it is possible to solve equations (2) with respect to $u_n'(a)$ and $u_n'(b)$:

$$\begin{aligned} u_n'(a) &= R_{1a}(\alpha_{ij}, \beta_{ij})u_n(a) + R_{1b}(\alpha_{ij}, \beta_{ij})u_n(b), \\ u_n'(b) &= R_{2a}(\alpha_{ij}, \beta_{ij})u_n(a) + R_{2b}(\alpha_{ij}, \beta_{ij})u_n(b), \end{aligned} \tag{34}$$

where the constants R do not depend on n. Using equalities (34), we can rewrite inequality (25) in the following form:

$$\sum_{k=1}^{n} \lambda_k f_k^2 \leqslant \int_a^b \left[(f'(x))^2 + q(x)f^2(x)\right] dx + R_{2b}(\cdot)\sigma_n^2(b,f) +$$
$$+ (R_{2a}(\cdot) - R_{1b}(\cdot))\sigma_n(a,f)\sigma_n(b,f) - R_{1a}(\cdot)\sigma_n^2(a,f). \tag{35}$$

Let the constants from (12) be denoted by $D(a,f)$ and $D(b,f)$. Therefore, there exists a number $n_2 \in \mathbb{N}$ such that for each entire number $n > n_2$ the estimate

$$\big|R_{2b}(\cdot)\sigma_n^2(b,f) + (R_{2a}(\cdot) - R_{1b}(\cdot))\sigma_n(a,f)\sigma_n(b,f) - R_{1a}(\cdot)\sigma_n^2(a,f)\big|$$
$$\leqslant \max\{|R_{ka}(\cdot)|, |R_{kb}(\cdot)| \mid k = 1,2\} \cdot \big(D(a,f) + D(b,f)\big)^2$$

holds. This estimate and (35) show how the estimate (19) should be modified.

The corresponding form of Proposition 3(c) can be "derived" from the modified Proposition 3(a) in the manner it was done in section 2.3.

4.2. On the proof of assertions (a) and (b). The necessary modifications of Propositions 3(a) and 3(c) being established, the proof of Theorem 2 has the same structure as the proof of Theorem 1. Actually, there is no essential difference between them, so we can omit the details.

4.3. On Remark 5. This remark is based on the following fact: If the boundary conditions (2) have one of the forms 1)–3), then instead of inequality (25) the inequality

$$\sum_{k=1}^{n} \lambda_k f_k^2 \leqslant \int_a^b \left[(f'(x))^2 + q(x) f^2(x)\right] dx$$

holds for every $n \in \mathbb{N}$. Hence, in this case, the corresponding Propositions 3(a) and 3(c) are valid, which implies that assertions of Theorem 1 hold.

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